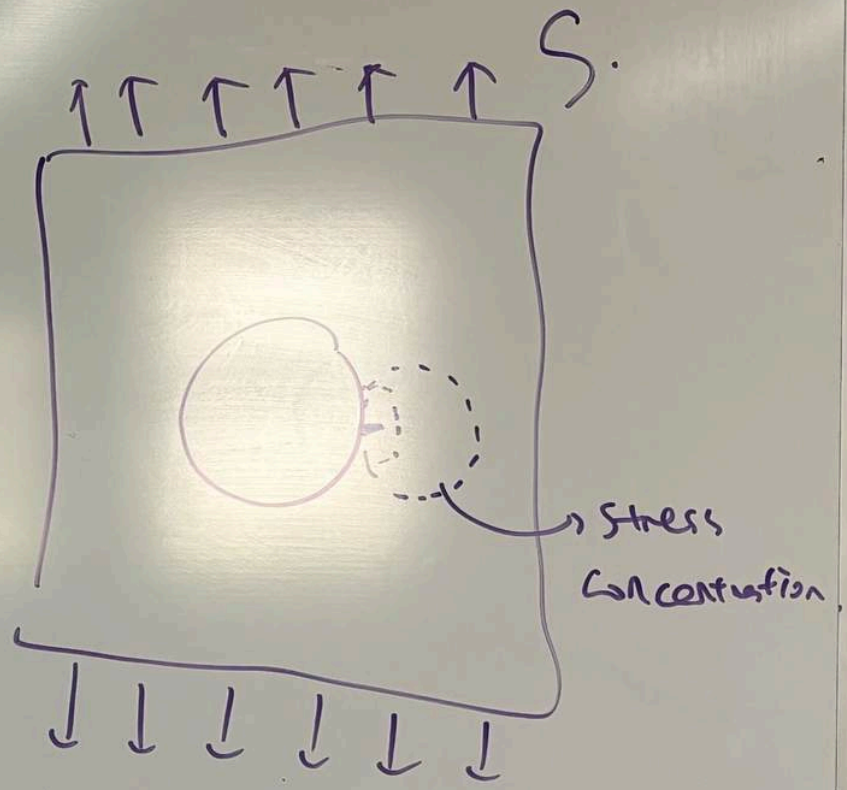
Problem Session - Final Review.

Mechanics: Elasticity & Inelasticity.

- Example of fatigue ✓
- Overview of fracture ✓
  - LEFM ...
  - Energy release rate
  - J-integral
- Overview of plasticity ✓
  - yield criteria
  - flow rule.

Continue example #2.



this is a hole ..

recall prev. deriv.

stress concentration factor = 3,

Cra

$K_I =$

in t

$K_I =$

$= 1.1$

ENGINEERING. consider mild steel.

$K_{IC} = 140\ MPa\sqrt{m}$. $C = 10^{-11}$ m/cycle. $m = 3$.

initial crack size: $a_0 = 0.5$ mm. $S = 75$ MPa.

$\sigma_Y = 250$ MPa.

$K_{IC} = 1.122 \times 3 \times S\sqrt{\pi a_c}$ ← solve for a_c

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{1.122 \times 3 \times S}\right)^2 = \frac{1}{\pi}\left(\frac{140}{1.122 \times 3 \times 75}\right)^2 = \boxed{0.098\,m}$$

Using the solved a_c ... solve for N.

$$N = \frac{1}{\left(\frac{3}{2}-1\right)10^{-11}\cdot(1.122 \times 3 \times 75)^3\,\pi^{3/2}}\left[(0.0005)^{-\frac{1}{2}} - (0.098)^{-\frac{1}{2}}\right]$$

$$= \frac{1}{4.479 \times 10^{-4}}(4.472 - 3.19)$$

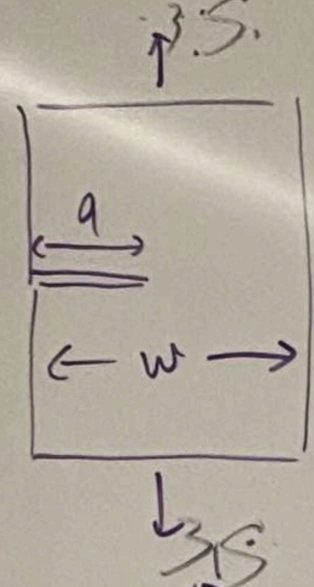
$$= 9.27 \times 10^4 \approx 10^5 \text{ cycles}$$

- midterm pickup
- practice final
- additional OH.

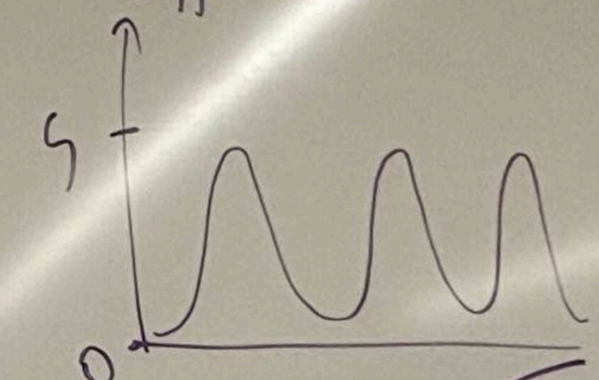
$\boxed{Q.}$

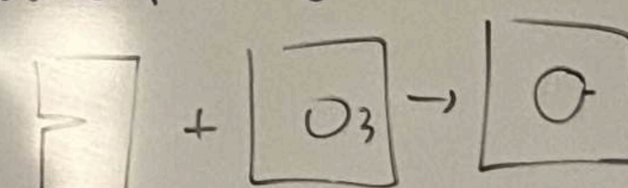
Crack is loaded by local stress. $3\sigma_{yy}^A$

σ_{yy}^A : oscillates between $0 \longleftrightarrow S$.



relate stress intensity factor. with crack size. consider. Single edge. notched. specimen under tension .....





$K_I = 1.122 \times 3.\ \sigma_{yy}^A\sqrt{\pi a}$

$\Delta K = 1.122 \times 3S\sqrt{\pi a}$

$$K_I = \frac{P}{B\sqrt{W}}\left[\frac{\sqrt{2\tan\left(\frac{\pi a}{2W}\right)}}{\cos\left(\frac{\pi a}{2W}\right)}\left(0.752 + 2.02\left(\frac{a}{W}\right) + 0.37\left(1-\sin\frac{\pi a}{2W}\right)^3\right)\right]$$

$= 2 + 7 - 3 - 5 \div (\cdot 2 \times 3) \div 3..$

$= 9$

in the limit of $w \to \infty$

$$K_I = \frac{P}{B\sqrt{W}}\sqrt{\frac{\pi a}{W}}\,(0.752 + 0.37).$$

$$= \frac{P}{BW}\sqrt{\pi a}\cdot 1.122$$

$$= 1.122\,3\sigma\sqrt{\pi a} \qquad \sigma = \frac{P}{BW}$$

$$\frac{da}{dN} = C(\Delta K)^m$$

$$= C(1.122 \times 3S)^m\,(\pi a)^{m/2}$$

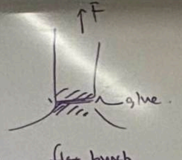
Crack growth per cycle.

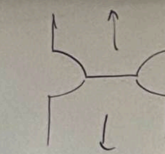
$$N = \frac{1}{\left(\frac{m}{2}-1\right)\cdot C(1.122\times 3S)^m\,\pi^{m/2}}\left(a_0^{-\frac{m}{2}+1} - a_c^{-\frac{m}{2}+1}\right)$$

fracture mechanics

Contact problem.

Surface Green function.





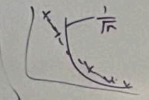
$$\frac{du_0}{dx} = \frac{K+1}{4\pi\mu}\int_{-c}^{c} \frac{P_y(x')}{x-x'}\,dx' \quad \text{flat punch}$$

integral equation. Sol'n.

$$P_y(x) = -\frac{1}{\pi^2\sqrt{c^2-x^2}}\int_{-c}^{c} \frac{\sqrt{c^2-x'^2}\,\frac{4\pi\mu}{k+1}\,\frac{du_0(x')}{dx'}}{x-x'}\,dx' + \frac{F}{\pi\sqrt{c^2-x^2}}.$$

from flat punch. contact.

$$P_y \sim \frac{F}{\pi}(2cr)^{-1/2} \rightarrow \boxed{\sigma_{yy} \propto \frac{1}{\sqrt{r}}}$$



Wedge & Notch

Trial sol'n: $\phi = r^{2}(A_1\cos 2\theta + A_2 + A_3\sin 2\theta + A_4\theta)$.

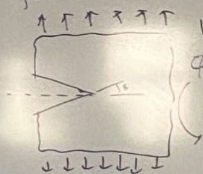
$$\sigma_{rr} = -2A_1\cos 2\theta + 2A_2 - 2A_3\sin 2\theta + 2A_4\theta$$

$$\sigma_{r\theta} = 2A_1\sin 2\theta + 0 - 2A_3\cos 2\theta - A_4$$

$$\sigma_{\theta\theta} = 2A_1\cos 2\theta + 2A_2 + 2A_3\sin 2\theta + 2A_4\theta.$$

$$\omega = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

formulate notch problem.



William's sol'n. $(n = \lambda - 1)$.

$$\phi = r^{\lambda+1}\{A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta\}.$$

... σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$. $\omega \to \infty$

Symmetric & anti-symmetric part (loading).

$\det(M_1) = 0$ — symmetric

$\det(M_2) = 0$. — anti-symmetric.

notch → crack $\theta = \pi$, $\theta = -\pi$ → $\lambda\sin 2\alpha \pm \sin 2\lambda\alpha = 0$.

$\alpha = 2\pi$ → $\sin 2\alpha = 0$

$\sin 2\pi\lambda = 0$

$\lambda = 0, \frac{1}{2}, \cancel{1}, \frac{3}{2}$. nonsingular

$\sigma \sim \frac{1}{\sqrt{r}}$

Slit-like crac…

$P_y(x) = -\frac{1}{\pi^2}$ …

... algebra

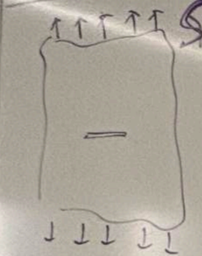
$P_y(x) = \frac{A + \ldots}{\sqrt{1-\ldots}}$

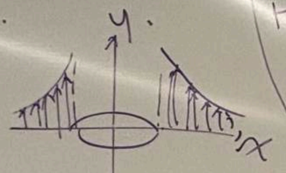
$\sigma_{yy}(x, y=0) =$ …

$x = a + r$ (…

$\sigma_{yy} \sim \frac{Sa}{\sqrt{2ar}} =$ …

Slit-like crack





Singular integral eqn.

$$p_y(x) = -\frac{1}{\pi^2}\frac{(x-a)^{1/2}}{(x+a)^{1/2}}\left[\int_{-\infty}^{-a} + \int_{a}^{+\infty}\right]\frac{(x'+a)^{1/2}}{(x'-a)^{1/2}}\frac{q(x')}{x-x'}\,dx' + \frac{Ax+B}{(x+a)^{1/2}(x-a)^{1/2}}.$$

... algebra

$$p_y(x) = \frac{A+B/x}{\sqrt{1-(a/x)^2}}.$$

$$\sigma_{yy}(x, y=0) = \frac{S\cdot|x|}{\sqrt{x^2-a^2}}$$

$x = a + r \quad (r \to 0^+)$

$$\sigma_{yy} \sim \frac{Sa}{\sqrt{2ar}} = S\sqrt{\frac{a}{r}}\frac{1}{\sqrt{r}}$$

recall wedge & notch.

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \quad \cdots \quad K_I = S\sqrt{\pi a}.$$

enthalpy: $H = E - \Delta W_{LM}$

... (l.e.)

$$H = \underbrace{\int_\Omega \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\,dV}_{\text{elastic strain energy.}} - \int_{S_t} T_j u_j\,dS.$$

$$\begin{cases} E = \frac{1}{2}\sigma_{yy}^A \varepsilon_{yy} V. \\ \Delta W_{LM} = (\sigma_{yy}^A A)\cdot(\varepsilon_{yy} L) = \sigma_{yy}^A \varepsilon_{yy} V. \\ H = \frac{1}{2}\sigma_{yy}^A\varepsilon_{yy}V - \sigma_{yy}^A\varepsilon_{yy}V = -\frac{1}{2}\sigma_{yy}^A\varepsilon_{yy}V = -E. \end{cases}$$

COD. $d(x) = \frac{2(1-\nu)}{\mu} Sa\sqrt{1-\left(\frac{x}{a}\right)^2}.$

enthalpy change. prop to app. stress S.

$$\Delta H = -\frac{1-\nu}{2\mu}S^2\pi a^2 \quad \text{(plane strain)}.$$

driving force. crack prop.

$$f_{el} = -\frac{\partial \Delta H}{\partial(2a)} = \frac{\pi(1-\nu)}{2\mu}S^2 a = \frac{1-\nu}{2\mu}K_I^2$$

$\frac{\partial H}{\partial c}$ γ_s = ... $> a$ (G = ...)

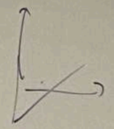
$\frac{\partial \gamma_s}{\partial c}$ = ... $< a$

Griffith Criteria

slit-like crack

$$\Delta G = \Delta H + \gamma_s \cdot 2 \cdot 2a.$$

$$\Delta G = -\frac{1-\nu}{2\mu}S^2\pi a^2 + 4\gamma_s a.$$



$$f_{tot} = \frac{\pi(1-\nu)}{2\mu}S^2 a - 2\gamma_s. \quad \cdots \text{solve } f_{tot} = 0.$$

$$a_c = \frac{4\mu}{\pi(1-\nu)}\frac{\gamma_s}{S^2}. \quad S_c = \sqrt{\frac{4\mu\gamma_s}{\pi(1-\nu)a}}$$

(plane strain)

Energy release rate

mode I, mode II, mode III.

$$G = -\frac{\partial(\Delta H)}{\partial(2a)}$$

$$= \frac{\pi(1-\nu)}{2\mu}(\sigma_{yy}^A)^2 a$$

$$G = \frac{\pi}{E'}(\sigma_{yy}^A)^2 a.$$

Plasticity review

displacement: $\underline{u} = \underline{x} - \underline{X}$.

Strain: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$,

Stress: $T_j = \sigma_{ij} n_i$

Equilibrium. $\sigma_{ij,i} + F_j = 0$.

for strain: $\varepsilon_{ij} = \underline{\underline{\varepsilon_{ij}^{el}}} + \varepsilon_{ij}^{pl}$.

$\sigma_{ij} = C_{ijkl}\, \underline{\underline{\varepsilon_{kl}^{el}}}$

hydrostatic stress: $\bar{\sigma} = \frac{1}{3}\sigma_{ii}$.

yield condition: $f(\{\sigma_{ij}\}) = 0$.

original stress invar. I_1, I_2, I_3.

Solve eigen-eqn.

$\det(\underline{\underline{S}} - \hat{\lambda}\,\underline{\underline{I}}) = \left| \ \cdots \ \right| = 0$

$\hat{\lambda}_1^3 + J_1 \hat{\lambda}^2 + J_2 \hat{\lambda} + J_3 = 0$

$J_1 = 0 \quad J_3 = \det(S_{ij}) \quad J_2 = \frac{1}{2} S_{ij} S_{ij}$.

$f(J_2) = J_2 - k^2 = 0$.

$\boxed{\text{EPP}: \dot{J}_2 = 0}$

Overall summary

$$\dot{\varepsilon}_{ij} \begin{cases} \dot{\bar{\varepsilon}} = \frac{1}{3}\dot{\varepsilon}_{ii} \rightarrow \dot{\bar{\sigma}} = 3k\dot{\bar{\varepsilon}} \\ \dot{e}_{ij} = \dot{\varepsilon}_{ij} - \dot{\bar{\varepsilon}}\delta_{ij} \end{cases}$$

$$\sigma_{ij} \begin{cases} \bar{\sigma} = \frac{1}{3}\sigma_{ii} \\ S_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij} \end{cases}$$

$\dot{W} = S_{ij}\dot{e}_{ij}$

$\dot{S}_{ij} = 2\mu\left(\dot{e}_{ij} - \frac{\dot{W}}{2k^2} S_{ij}\right)$

$\dot{\sigma}_{ij} = \dot{S}_{ij} + \dot{\bar{\sigma}}\delta_{ij}$

recall $K_I = \sigma_{ff}\sqrt{\pi a} \rightarrow G = \frac{K_I^2}{E'}$ ... mode I

General crack case: $G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{E'}$

Comment. $G \leftarrow \cdots \frac{\partial H}{\partial 2a}$

$2a$

J-integral

$J = \int_{\Gamma} w \, dy - T \cdot \frac{\partial u}{\partial x} \cdot ds$

$J = G$

$= \int_{V} \frac{\partial w}{\partial x_i} \, dV \cdot - \int_{S} T_j \frac{\partial u_j}{\partial x_i} \cdot dS$

Example 1

$J = 0$

Example 2

Example 3

$\Gamma \quad J = \cdots = \frac{K_I^2}{E'}$

Example 4

Γ

Elastic-Plastic fracture mechanics

$\frac{\sigma}{\sigma_Y}$

r

$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$

$\sigma_Y = \frac{K_I}{\sqrt{2\pi r_y}}$

$r_p = 2r_y$

$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_Y}\right)^2$

fracture criteria: $J = J_c$

- plastic yielding. change K_I: $K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a_{eff}}{W}\right)$. $a_{eff} = a + r_y$

HRR solution $\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$ strain-hardening exp.

▷ Strip-yield model ...

Irwin's approach.

$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 = 0.318\left(\frac{K_I}{\sigma_Y}\right)^2$

- energy release rate.

$G = J = \int_{\Gamma} w \, dy - T \frac{\partial u}{\partial x} ds$

...

$= \sigma_Y \cdot \delta$

$G = J = \sigma_Y \cdot \delta$

fracture. $\sigma_Y \cdot \delta_c$

fracture COD