Problem Session - Midterm Review

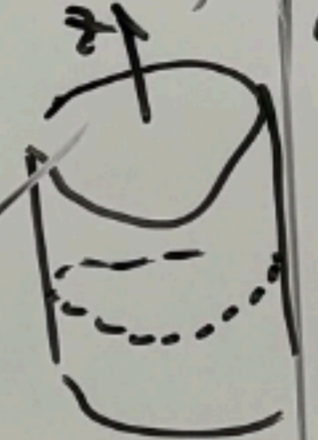
Basic equations.

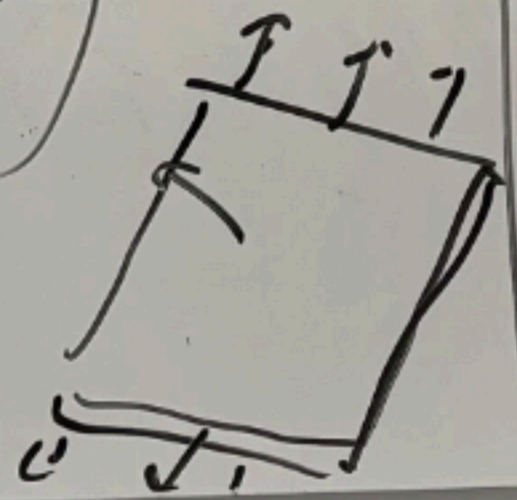
What have we learned so far?

1. Basic equations. (elasticity) ✓
*2. Beam problems (stress function). ✓
*3. Surface Green function. ✓
*4. Polar coordinates. (Michell table) ✓
5. Contact problem. (scift)
6. Wedge & Crack ✓ ...

Biharmonic

$\nabla^4 \phi = 0 \quad \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$





What are the eqs?

$$\begin{cases} \sigma_{ij,i} + F_j = 0 & \text{... equilibrium.} \\ \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 & \text{...} \\ \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} & \text{sym.} \\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) & \end{cases}$$

3D

2D ← Simplification → assumptions,

plane strain $\begin{cases} \varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz} \\ \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{zz}. \end{cases}$

plane stress, $\begin{cases} \varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{xy} = -\frac{\nu}{E}\sigma_{xx} + \frac{1}{E}\sigma_{yy} \end{cases}$

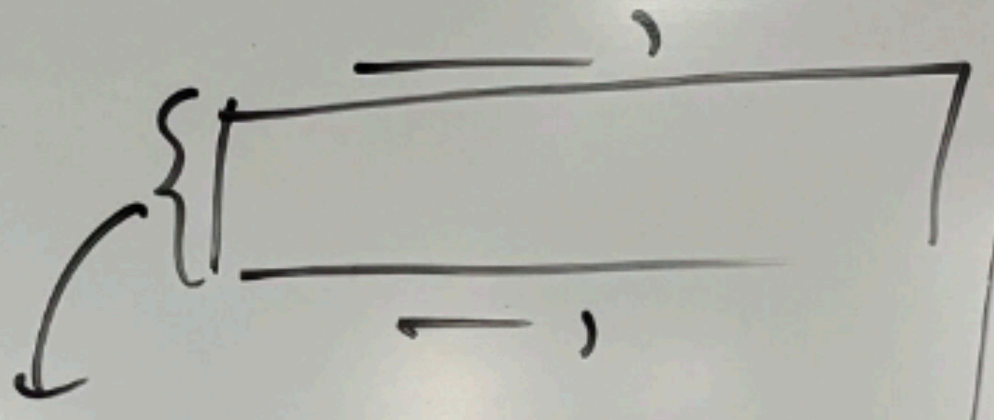
K → Kolosov const

Beam problem — $\boxed{\text{Stress function approach}}$

$$\begin{cases} \sigma_{xx} = \phi_{,yy} \\ \sigma_{yy} = \phi_{,xx} \\ \sigma_{xy} = -\phi_{,xy}. \end{cases}$$

automatically satisfies. → $\nabla^4 \phi = 0.$

equilibrium & compatibility



→ impose weak B.C.s on the short side

→ Strong B.C.s on the longitudinal direction.

Pg #3: lec. #8

$\phi(x,y) = \cos\lambda x \, [A' \cosh\lambda y + D' y \sin\lambda y]$

$\phi(x,y) = \cos\lambda x \, [B' y \cosh\lambda y + C' \sinh\lambda y].$

$\phi(x,y) = \sin\lambda x \, [A' \cosh\lambda y + D' y \sinh\lambda y]$

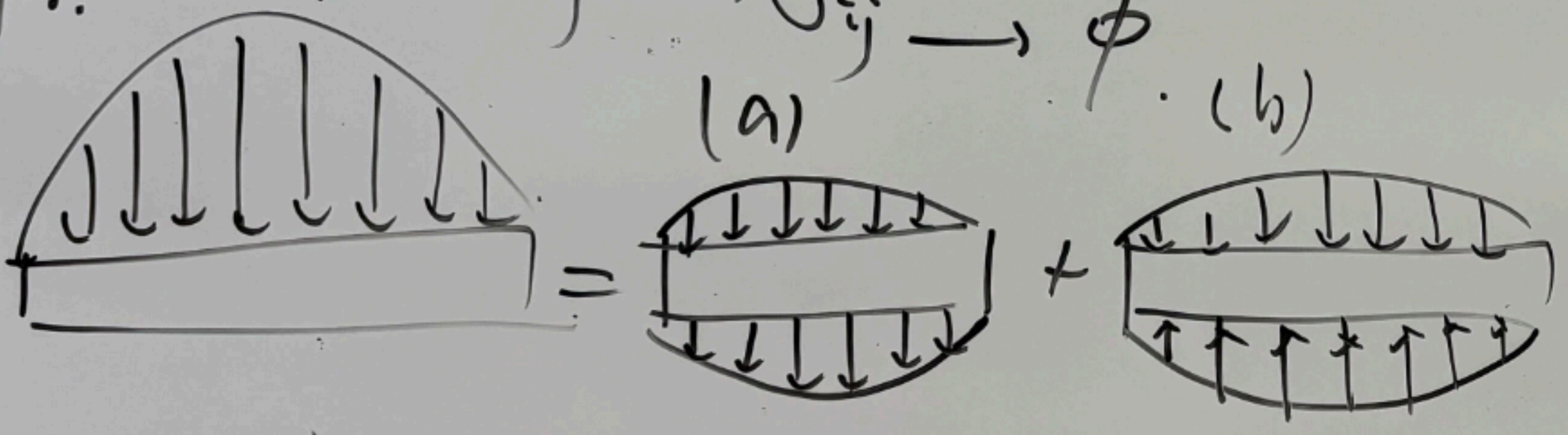
$\phi(x,y) = \sin\lambda x \, [B' y \cosh\lambda y + C' \sinh\lambda y]$

Ways to construct stress function. → Fourier expansion

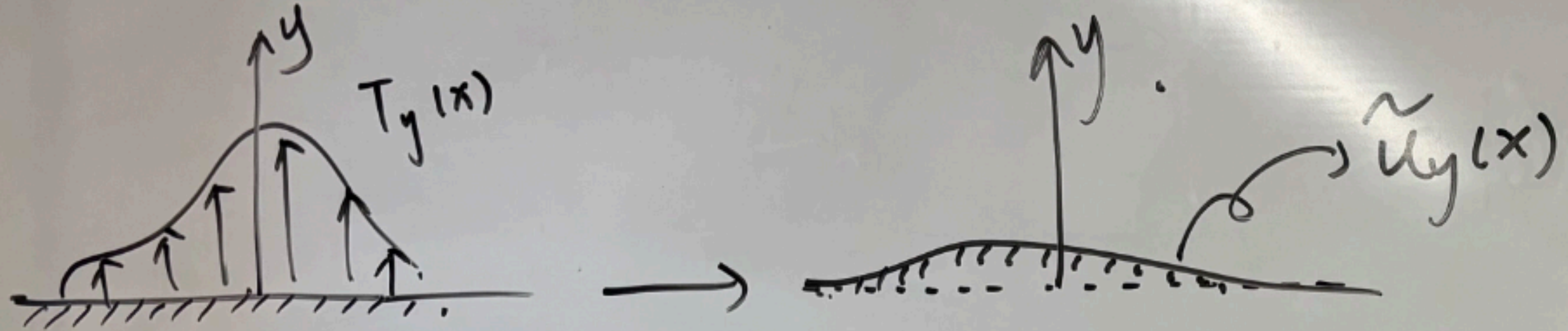
Fourier solution

Symmetry!! → Superposition.

$u_i \longrightarrow \varepsilon_{ij} \longrightarrow \sigma_{ij} \longrightarrow \phi$



Surface Green function



"Surface displacement"

$$\begin{cases} \tilde{u}_y(x) = \int_{-\infty}^{+\infty} T_y(x')\, G^S_{yy}(x-x')\, dx' \\ \tilde{u}_x(x) = \int_{-\infty}^{+\infty} T_y(x')\, G^S_{xy}(x-x')\, dx'. \end{cases}$$

Polar coordinates | $\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}$

Pg. #2 for lec #11

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_{\theta\theta} = \partial^2\phi / \partial r^2$$

Biharmonic eqn.

$$\nabla^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\cdot\ \cdot)\phi = 0.$$

Pg #6 for lec. #11

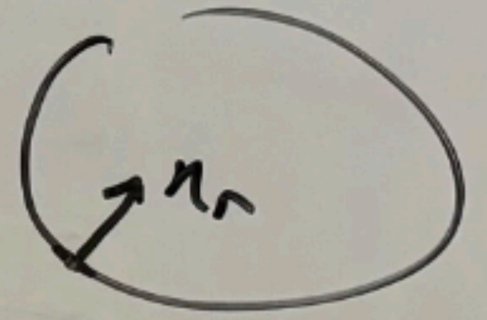
$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)$$

Logic for solving problems ...

disp.	strain.	stress	traction
$u_r(r,\theta)$	$\varepsilon_{rr}(r,\theta)$	$\sigma_{rr}(r,\theta)$	T_r
$u_\theta(r,\theta)$ $\dashrightarrow$	$\varepsilon_{r\theta}(r,\theta)$ $\rightarrow$	$\sigma_{r\theta}(r,\theta)$ $\rightarrow$	T_θ
	$\varepsilon_{\theta\theta}(r,\theta)$.	$\sigma_{\theta\theta}(r,\theta)$	



local. per

(usually around

$\phi^{(1)} \longrightarrow \sigma_{ij}^{(1)} \longleftarrow$

Go

{ co

k

ε

for lec. #1.

$\frac{\partial u_r}{\partial r}$

$= \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$

$= \left(\frac{1}{2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right)$

Governing equations.

- constitutive laws. Eqn. (46)
- kinematics. ... Eqn. (37).
- Equilibrium ... Eqn. (55).

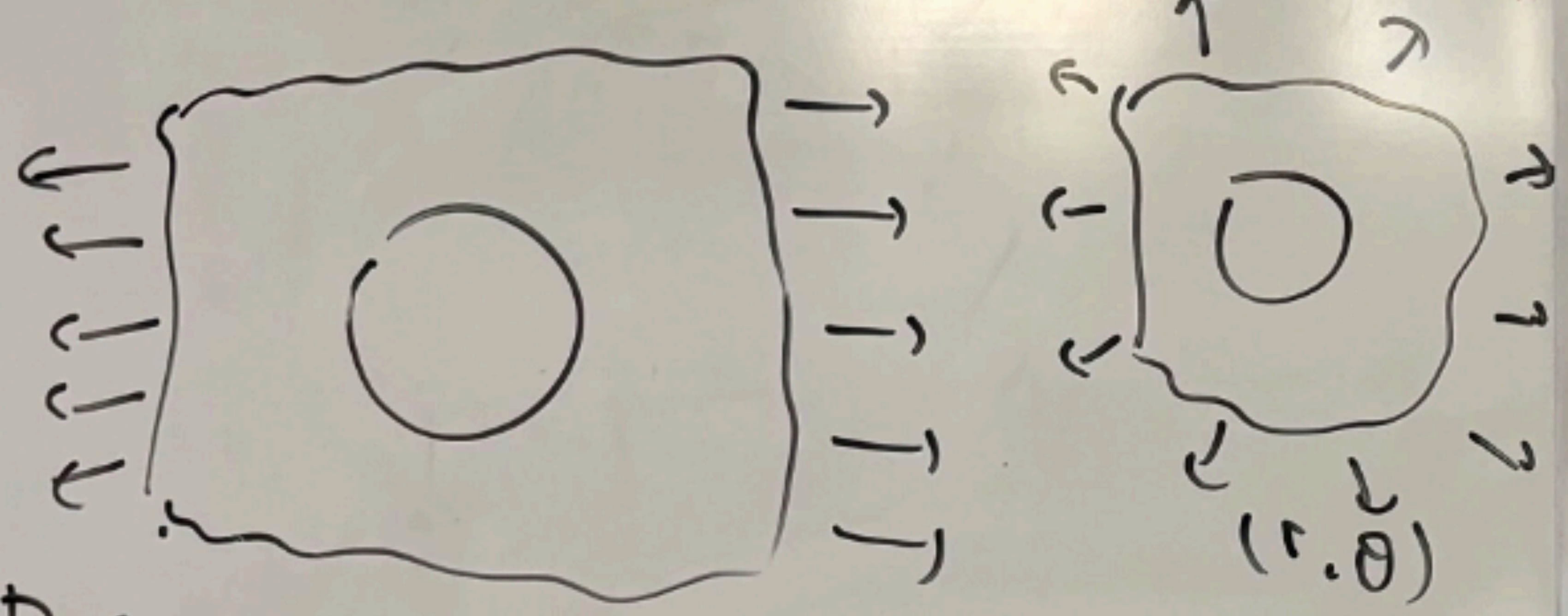
$\to u_i,\ \varepsilon_{ij},\ \underline{\sigma_{ij}}$ ....

$\downarrow$

Stress concentration factor.

for solving problems ...

strain,	stress	traction
$\varepsilon_{rr}(r,\theta)$	$\sigma_{rr}(r,\theta)$	T_r
$\varepsilon_{r\theta}(r,\theta) \to$	$\sigma_{r\theta}(r,\theta) \to$	T_θ
$\varepsilon_{\theta\theta}(r,\theta)$.	$\sigma_{\theta\theta}(r,\theta)$	



B.C.s $\sigma_{ij}(r \to \infty) = \ldots (x,y)$

$\sigma_{ij}^{(0)} \to \phi^{(0)}$.

$)\phi = 0.$

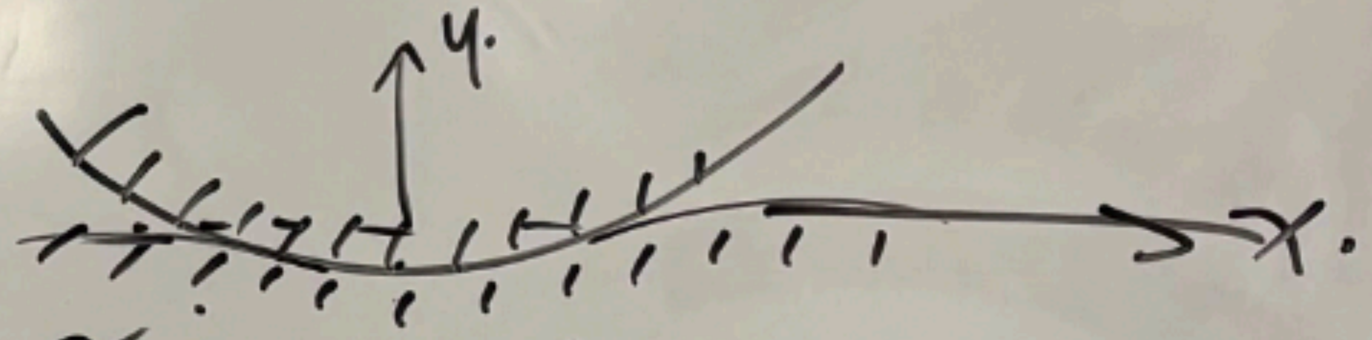
local perturbation (usually around circle)

$\phi^{(1)} \leftrightarrow \sigma_{ij}^{(1)} \leftrightarrow \varepsilon_{ij}^{(1)} \leftrightarrow u_i^{(1)}$

overall stress func.

$\phi = \phi^{(0)} + \phi^{(1)}$

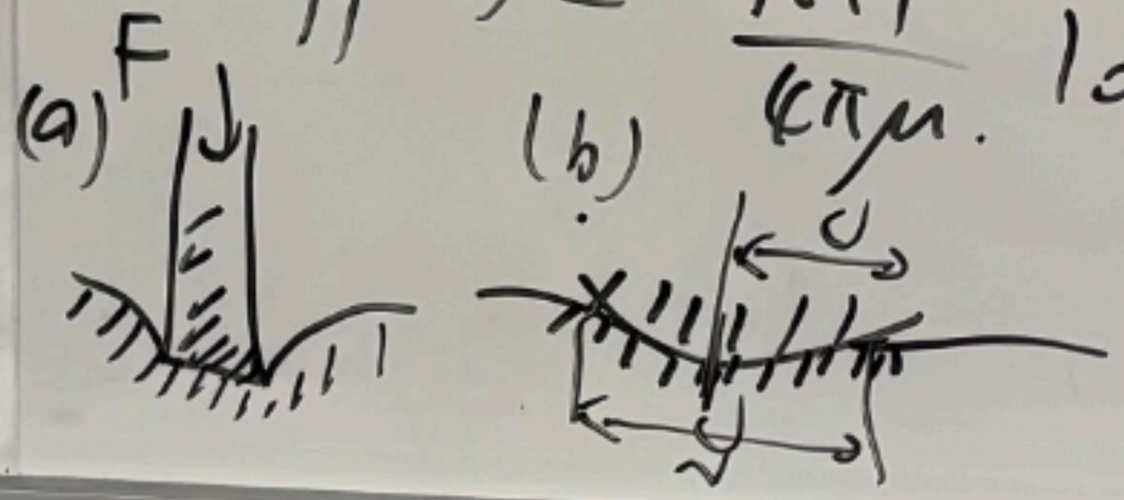
Contact problem



$$\tilde{u}_i(x) = \int_{-\infty}^{+\infty} -p_j(x')\, G^s_{ij}(x-x')\, dx'$$

$$G^s_{xy}(x) = \frac{\kappa - 1}{8\mu}\,\mathrm{sgn}(x).$$

$$G^s_{yy}(x) = -\frac{\kappa+1}{4\pi\mu}\,\log|x|$$



... See lecture notes for 3D

(a). $P_y(x) = \dfrac{F}{\pi\sqrt{c^2 - x^2}}$

(b) $P_y(x) = \dfrac{2F}{\pi c^2}\sqrt{c^2 - x^2}$

St. Venant's principle,

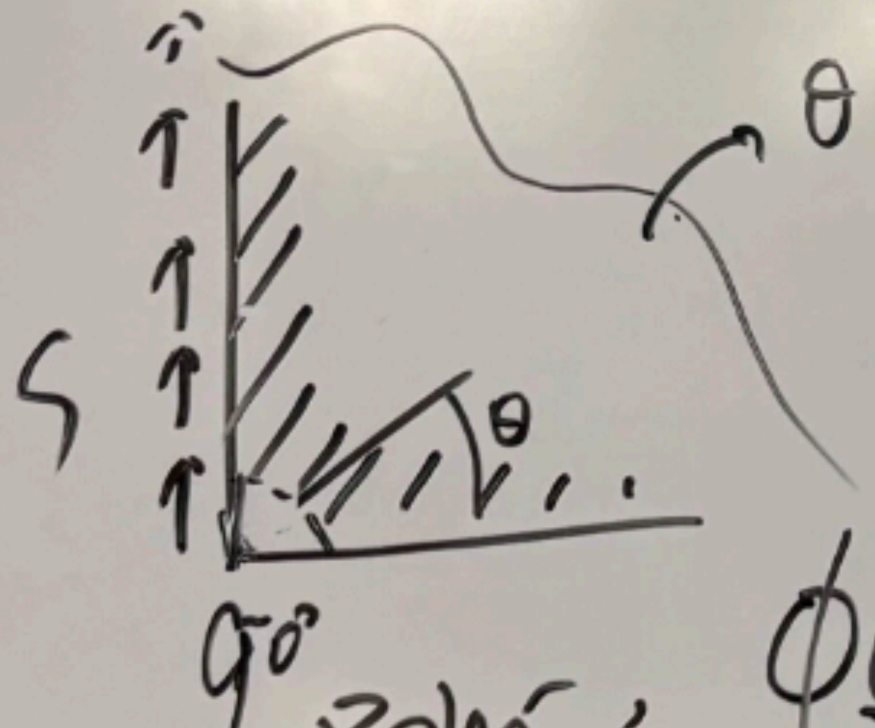
520-131

O.H. 3:30 PM

Sat. 2-4 PM.

Sol'n after 10:45 AM

Wedge & Notch → crack (fracture mechanics) ...



$\theta = \pi/2 : \sigma_{r\theta} = S.$ } Michell table.

$\phi = r^2(A_1\cos 2\theta + A_2 + A_3\sin 2\theta + A_4\theta)$ using B.Cs.

polar, $\phi(r,\theta) = S\left(-\dfrac{\pi r^2\cos 2\theta}{8} + \dfrac{\pi r^2}{8} + \dfrac{r^2\sin 2\theta}{4} - \dfrac{r^2\theta}{2}\right)$ algebra ...

Cartesian $\phi(x,y) = S\left(-\dfrac{\pi}{8}(x^2-y^2) + \dfrac{\pi}{8}(x^2+y^2) + \dfrac{xy}{2} + \dfrac{x^2+y^2}{2}\arctan\dfrac{y}{x}\right)$

→ $\sigma_{xy} = \sigma_{yx} = -Sy^2/(x^2+y^2)$ ... non physical

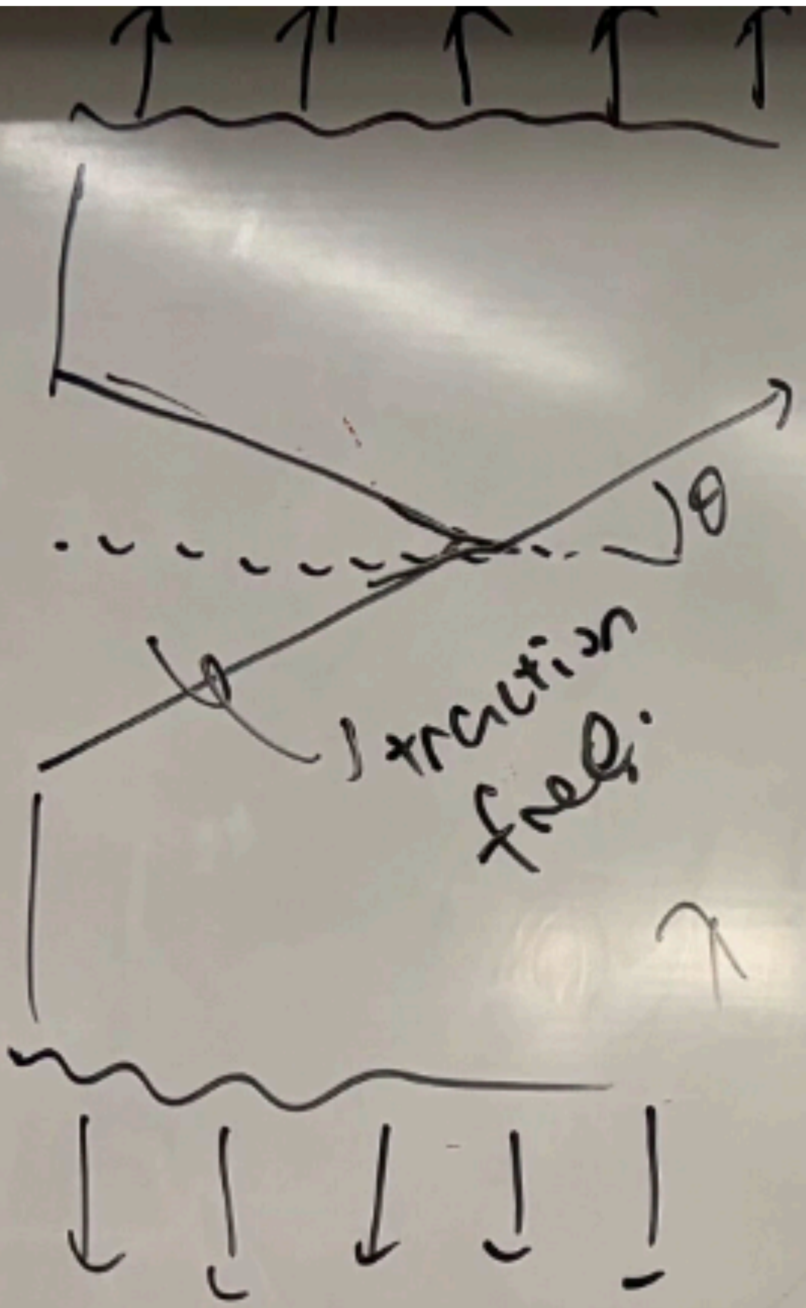
...

-tive factor.

s

$_{ij}(x-x')dx'$

e lecture notes for 3D



Michell table

"william's soln".

$$\phi = r^{n+2}\left[A_1\cos(n+2)\theta + A_2\cos n\theta + A_3\sin(n+2)\theta + A_4\sin n\theta\right].$$

$$\phi = r^{\lambda+1}\left[A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta\right].$$

some algebra

$$\lambda\sin 2\alpha \pm \sin 2\lambda\alpha = 0.$$

Stress singularity

$$\sigma \sim r^{\lambda-1}$$

$r^{-1/2}$,

general.

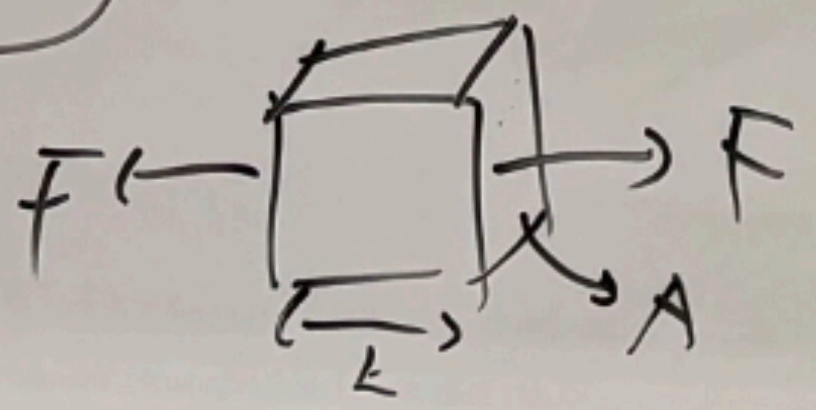
$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$$

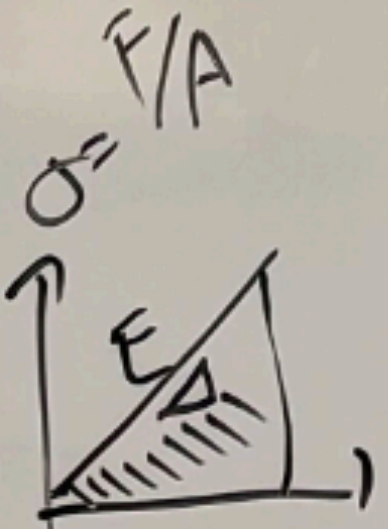
$\lambda = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$\lambda = 1, \frac{3}{2}, \dots$ non-singular stress field. (stress $\not\to \infty$ when $r \to 0$)

$\lambda = 0$. why?

$\frac{1}{2}$ ✓





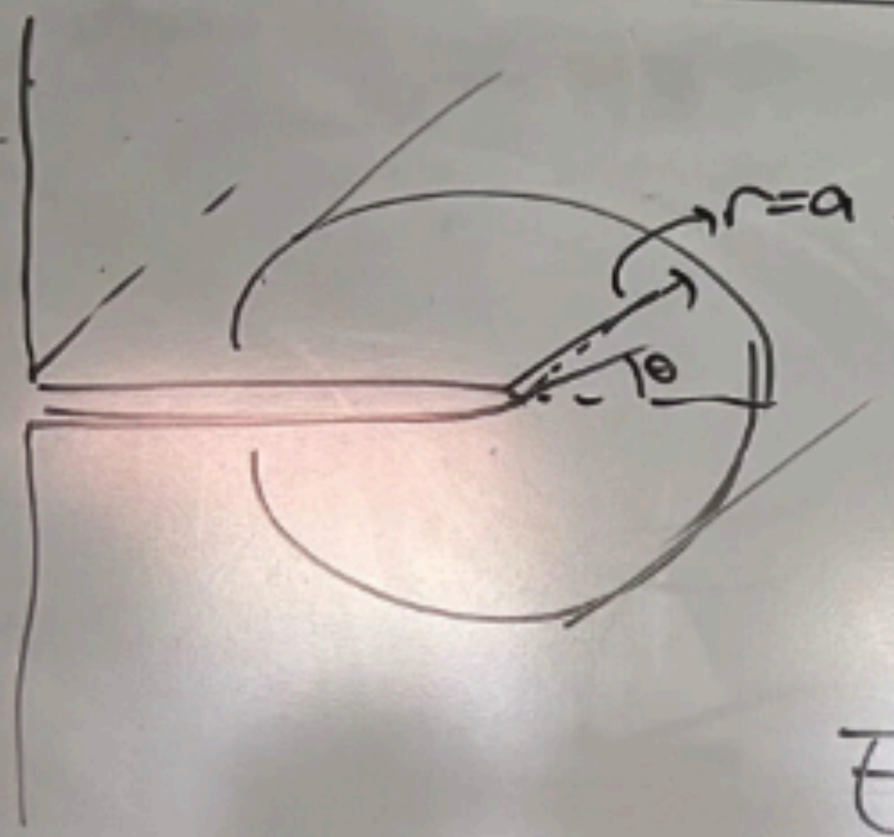
$\varepsilon = \delta / L$

Elastic strain energy.

$$E_{el} = \Delta W = \int_0^{\delta} F\,d\delta = \int_0^{\varepsilon} (\sigma' A) L\,d\varepsilon'$$

: algebra

$$= A \cdot L \cdot \frac{1}{2}\sigma\varepsilon$$



total elastic energy stored in cylindrical region of r=a. (around circle)

$$E_{el} = \int_0^a r\,dr \int_{-\pi}^{\pi} d\theta \,.$$

if $\lambda = 1/2$.

$\sigma \sim 1/\sqrt{r}$

$$w \sim \frac{1}{r}. \quad E_{el} \propto \int_0^a r\cdot\frac{1}{r}\,dr = a. \quad \boxed{\text{finite}}$$

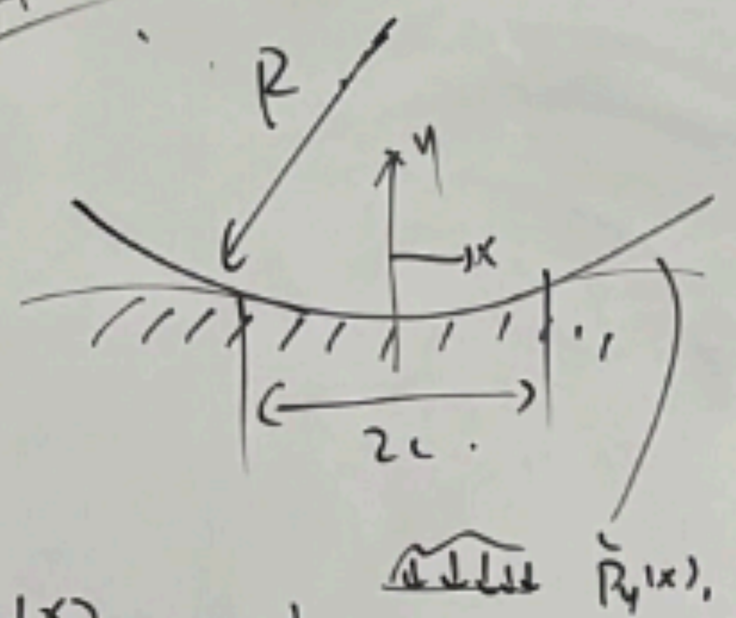
if $\lambda = 0$

$\sigma \sim 1/r$

$$w \sim \frac{1}{r^2} \quad E_{el} \propto \int_0^a \frac{1}{r}\,dr = \ln r\Big|_0^a \to \infty.$$

Example

(a)



Parabolic shape.

$u_0(x) = x^2/2R$

$$g(x) = \frac{du_0}{dx} = \frac{4\pi\mu}{k+1}\left(\frac{x}{R}\right).$$

$$P_y(x) = \frac{-1}{\pi\sqrt{c^2-x^2}} \cdot \text{P.V.} \int_{-c}^{+c} \frac{\sqrt{c^2-x'^2}}{x-x'}\,\frac{4\pi\mu}{k+1}\left(\frac{x'}{R}\right)dx' + \frac{F}{\pi\sqrt{c^2-x^2}}.$$

(principal value) check lecture notes

... apply change of var.

... algebra.

$$(*) \quad P_y(x) = \frac{4\mu}{(k+1)R}\sqrt{c^2-x^2} + \underbrace{\left(\frac{F}{\pi} - \frac{2\mu c^2}{(k+1)R}\right)}\frac{1}{\sqrt{c^2-x^2}}$$

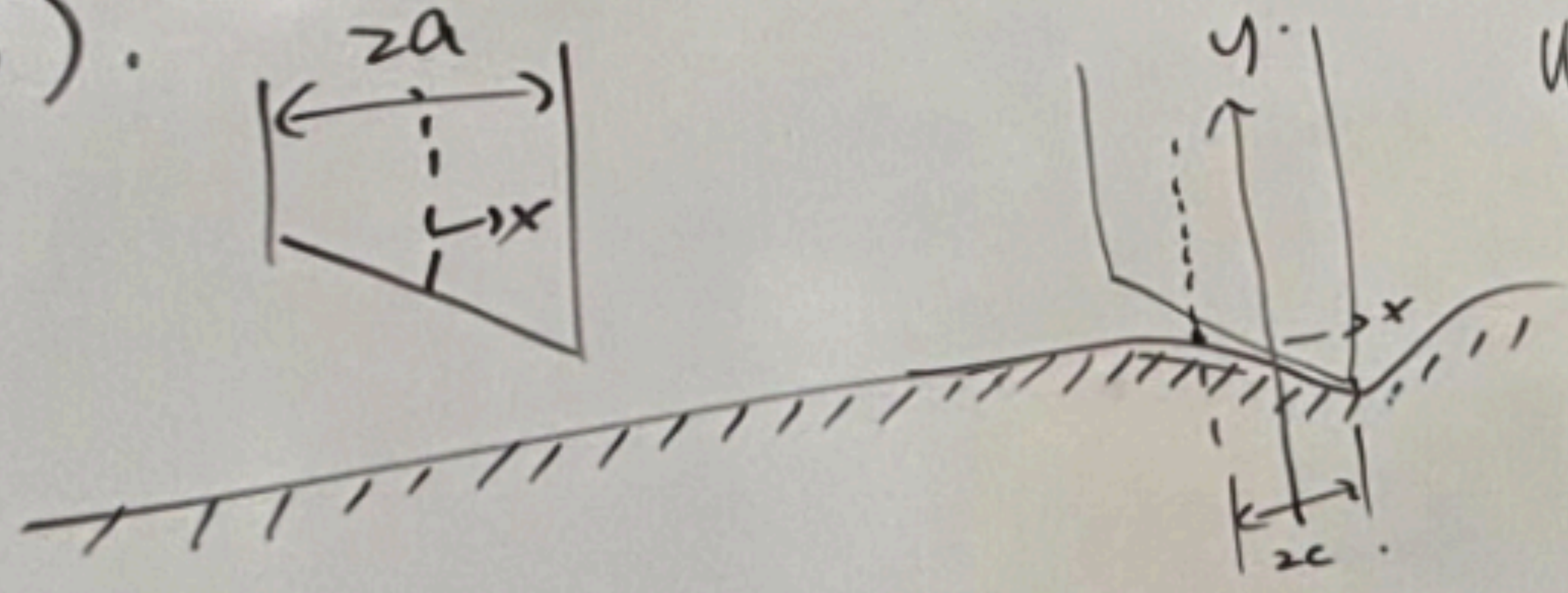
singularity

$$F = \frac{2\pi\mu c^2}{(k+1)R} \rightsquigarrow \sigma = 0.$$

total indentation force. $F = \int_{-c}^{+c} P_y(x)\,dx.$

$$P_y(x) = \frac{4\mu}{(k+1)R}\sqrt{c^2-x^2}$$

(b).



$u_0(x) = \beta(a-x)$

$\frac{du_0(x)}{dx} = -\beta.$

... algebra ...

recall eqn. (*)

$$P_y(x) = -\frac{1}{\pi^2\sqrt{c^2-x^2}} \; \text{P.V.} \left[\int_{-c}^{c} \cdots \frac{4\pi\mu}{k+1}(-\beta)\,dx' + \cdots\right] \quad (**)$$

Coordinate transformation.

$x_2 = x-(a-c)$ } eqn. (**).

$$P_y(x_2) = \frac{-1}{\pi^2\sqrt{c^2-x_2^2}} \cdot \text{P.V.} \left[\int_{-c}^{+c} \cdots \frac{4\pi\mu}{k+1}(-\beta)\,dx_2' \cdots\right].$$

See lecture note. #10. Pg. #3 eqn. (10)

Wikipedia. Cauchy principal value.

x)

...

$\because P_y(-c) = 0$

$\therefore \frac{F}{\pi} - \frac{4\mu\beta c}{(k+1)} = 0$

$$F = \frac{4\pi\mu\beta c}{k+1}.$$