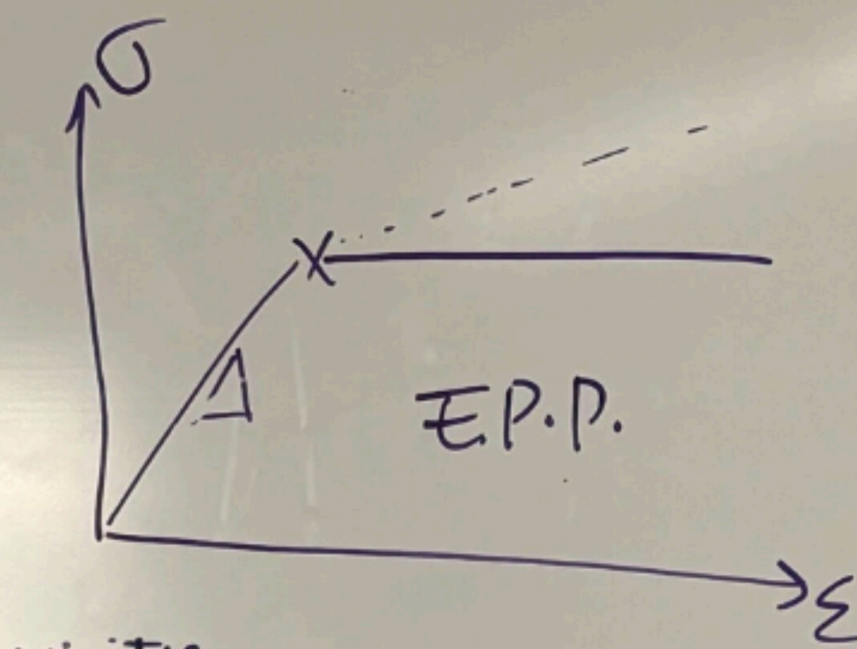
Problem Session

- flow rule ✓
- fracture mechanics ✓
- J-integral (1968) ✓
- summary of fracture ✓

general sol'n procedure of plasticity:

1. elastic response
2. yield condition
3. flow rule

$\left.\begin{matrix}\text{2. yield condition}\\ \text{3. flow rule}\end{matrix}\right\}$ plasticity.



1. Elastic response

bulk mod.

$$\bar{\sigma} = 3K\,\bar{\varepsilon}^{el}$$

$$G \;\; = \;\; \mu \qquad \bar{\varepsilon}^{el} \equiv \bar{\varepsilon}$$

$$s_{ij} = 2\mu\, e^{el}_{ij}$$

2. yield condition. $\sigma_{ij} = \bar{\sigma}\delta_{ij} + s_{ij}$

$$f(\{\sigma_{ij}\}) = 0 \;\Big|\; f(\{\sigma_{ij}\}) < 0 \quad f(\{\sigma_{ij}\}) = 0 \quad \cancel{f(\{\sigma_{ij}\}) > 0} \;\Big|$$

comm

▷ for

$\tilde{\lambda}$

fract

"Grif

→ me

evolu

enthal

comment no rate-dependence in this model

no strain-hardening.

▷ for J_2 plasticity. w/o strain-hardening.

$$\tilde{\lambda} = \frac{2\mu}{2k^2} \dot{W} \longrightarrow \dot{W} = S_{ij}\,\dot{e}_{ij} \qquad \varepsilon_{ij}$$

rate of work.
due to shape change

fracture mechanics

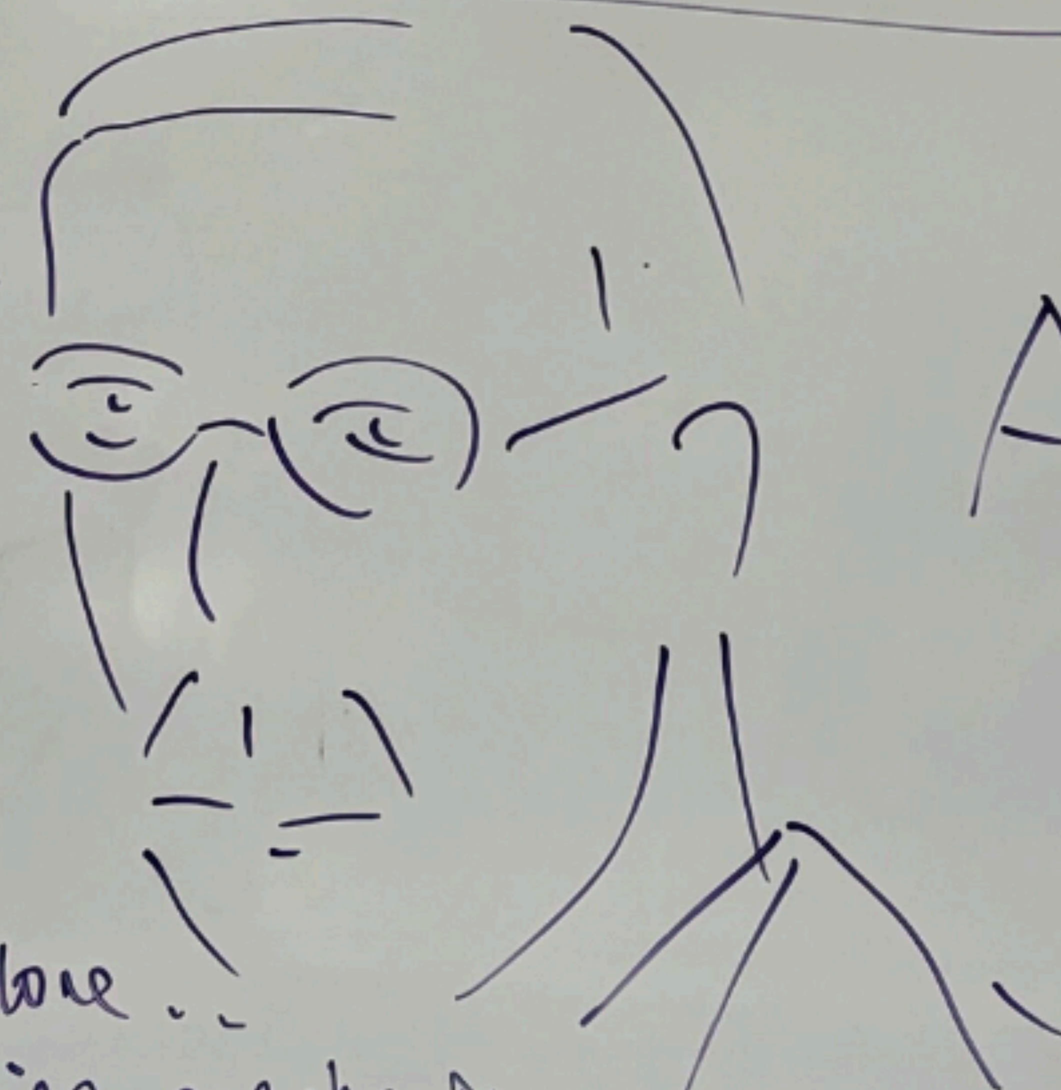
"Griffith approach" (no thermal reservoir).

→ mechanical systems tend to evolve in ways minimize enthalpy

enthalpy: $H = \hat{E} - \Delta W_{in}$

$\hat{E}$ ← internal energy

ΔW_{in} ← work done. loading mechanism

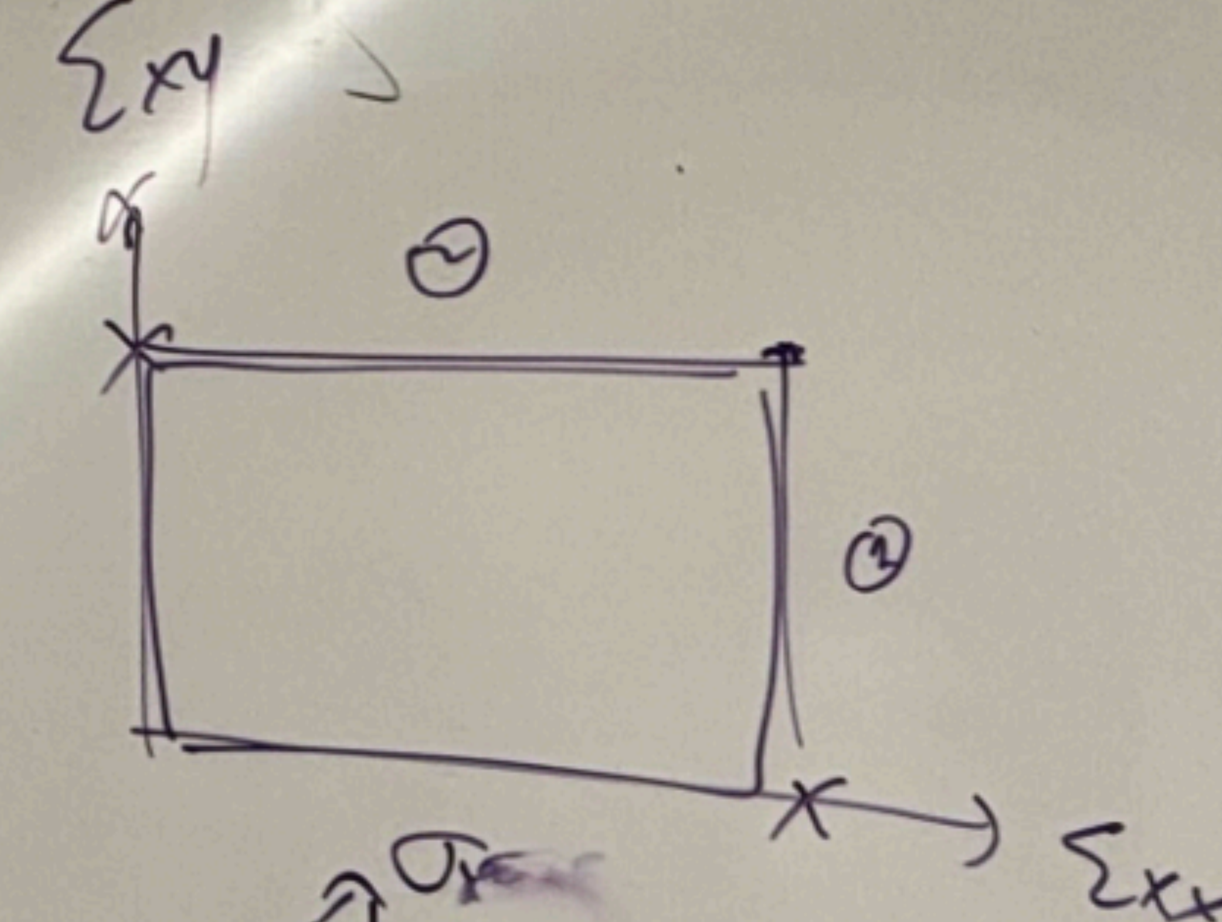


A.A. Griffith

... isotropic. no hydrostatic dependence.

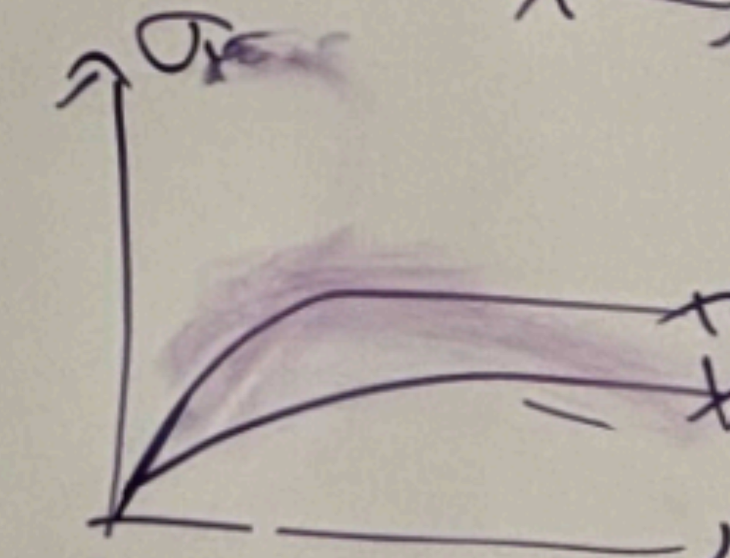
J_2 plasticity. $J_2 = \frac{1}{2} S_{ij} S_{ij}$

- Von Mises: $J_2 = k^2$.
- Tresca: $\sigma_1 - \sigma_3 = 2k_T$ ....



*** flow rule.

- plasticity is path-dependent. We need to think abt incrementally .... flow rule tells us the plastic strain increment.



Associative flow rule — metal — calc. plastic strain increm.

$$\dot{\varepsilon}^{pl}_{ij} = \frac{\hat{\lambda}}{2\mu} S_{ij}, \quad d\varepsilon^{pl}_{ij} = \frac{S_{ij}}{2\mu}\hat{\lambda}\, dt \rightarrow \varepsilon^{pl}_{ij} = \int d\varepsilon^{pl}_{ij} = \int \frac{1}{2\mu} S_{ij}(t)\hat{\lambda}(t)\, dt$$

$$H = \int_\Omega \underbrace{\frac{1}{2}\sigma_{ij}\varepsilon_{ij}}_{\text{Strain energy density (L.E.).}} d\mathcal{V} - \int_S \underbrace{T_j u_j}_{\text{work of tractions per unit area.}} dS.$$

integral over volume

integral over surface

Recall Hooke's law: $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$,

$\vdots$

$\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl}$ compliance.

internal energy

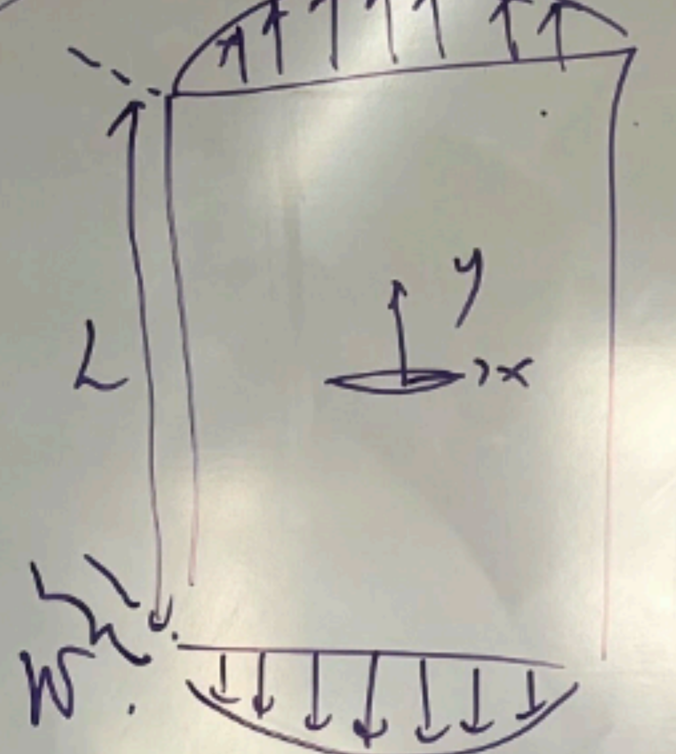
$$\hat{E} = \int_\Omega \frac{1}{2} S_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\, d\mathcal{V}.$$

isotropic, l.e. solid.

$$= \frac{1}{2}\int_\Omega \left\{\frac{1}{E}\left[\sigma_{ij}\sigma_{ij} - \nu(\sigma_{kk}\sigma_{ii} - \sigma_{ij}\sigma_{ij})\right]\right\} d\mathcal{V}$$

$$= \frac{1}{2E}\int_\Omega \left\{(1+\nu)\,\sigma_{ij}^2 - \nu\,\sigma_{kk}^2\right\} d\mathcal{V}$$

Example



plane strain.

$$\sigma_{yy} = -\frac{S}{b^2}(x^2 - b^2),$$

$$\hat{E} = \frac{1}{2}\int_\Omega \sigma_{ij}\varepsilon_{ij}\, d\mathcal{V}.$$

$$= \frac{WL}{2}\int_{-b}^{b} (\sigma_{yy}\varepsilon_{yy} + \cancel{\sigma_{zz}\varepsilon_{zz}}^{\,0})\, d\mathcal{V}.$$

Const. law: $\varepsilon_{yy} = \frac{1-\nu^2}{E}\sigma_{yy}$.

$$\hat{E} = \frac{WL}{2}\int_{-b}^{b} \frac{1-\nu^2}{E}\,\sigma_{yy}^2(x)\, dx$$

$$= \frac{WL(1-\nu^2)}{2E}\int_{-b}^{b} \frac{S^2}{b^4}(x^2 - b^2)^2\, dx$$

$\cdots$

$$= \frac{8WLS^2 b(1-\nu^2)}{15E}$$

$$u_y = \int_0^L \varepsilon_{yy}\, dy$$

$$= L\,\varepsilon_{yy},$$

$$W_{cm} = \int_A \sigma_{yy}\, u_y\, dA$$

$$= \int_{-b}^{b} W\,\sigma_{yy}(x)\, L\,\varepsilon_{yy}(x)\, dx$$

$\cdots$

$$= \frac{16WLS^2 b(1-\nu^2)}{15E}$$

$H = \hat{E} - \Delta W_{LM}$

$= -\dfrac{8WLS^2b(1-\nu^2)}{15E}$

$= -\hat{E}$.

*** linear elastic body w/o internal stress: $H = -\hat{E}$.

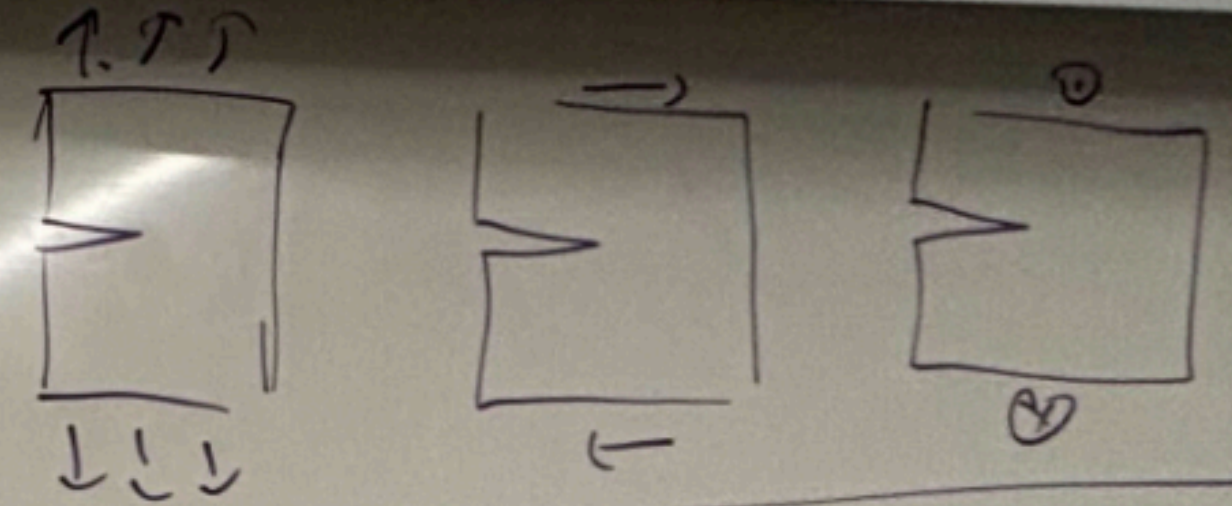
*** for a cracked body.

$H = H(a)$

→ $\dfrac{\partial H}{\partial a}$ tells us How much H is reduced when a increases.

We define $G = -\dfrac{\partial H}{\partial a}$ → strain energy release rate.

then for diff. modes of fracture.



$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \qquad E' = \begin{cases} \dfrac{E}{1-\nu^2} & \text{plane strain.} \\ E. & \text{plane stress} \end{cases}$$

$$= G_I + G_{II} + G_{III}.$$

" G contains the same information as K_I, K_{II}, K_{III}".

fracture criteria: $G \geq \Gamma$ or $K_I > K_{IC}$

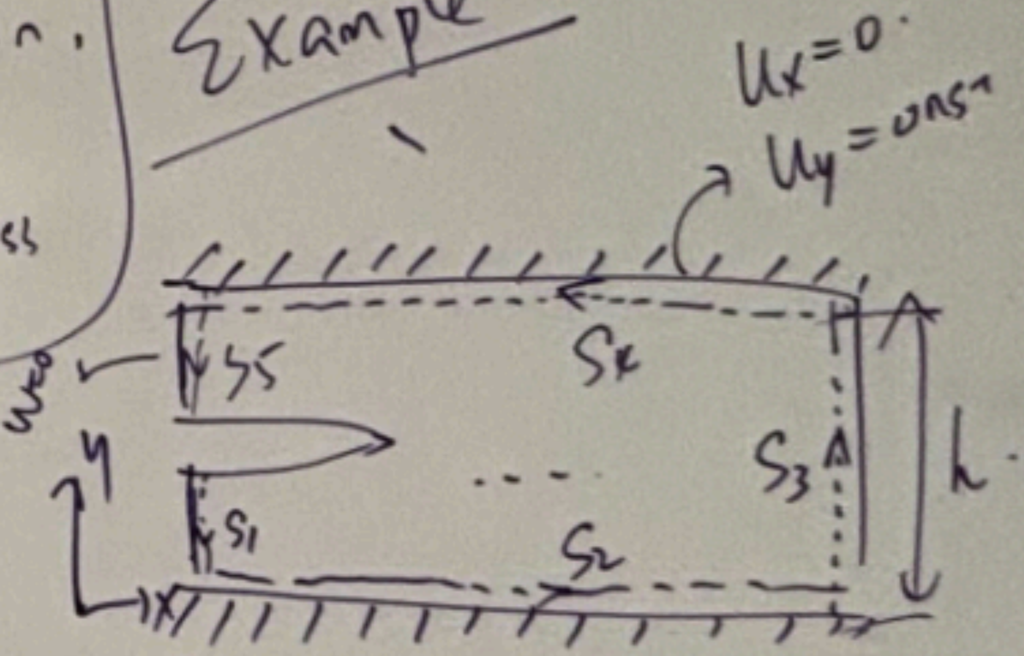
$K_{II} > K_{IIC}$

$K_{III} > K_{IIIC}$.

more than one of K_I, K_{II}, K_{III} is nonzero

"Mixed-mode" fracture

Example



$J_i = \int_S (w n_i - T_j u_{j,i})\, dS$

$J = G$

Example (Cont'd)

$$J(S_2) = \int_{S_2} W \cancel{dy}_{\swarrow 0} - \underset{0}{\underline{T}} \cdot \cancel{\frac{\partial u}{\partial x}} ds = 0$$

$$J(S_4) = 0$$

$$J(S_1) = \int_{S_1} \cancel{W\,dy}^{\nearrow 0} - \underline{T} \cdot \cancel{\frac{\partial u}{\partial x}}^{\nearrow 0} ds = 0$$

$$J(S_5) = 0$$

$$J(S_3) = \int_{S_3} W\,dy - \underline{T} \cancel{\frac{\partial u}{\partial x}}^{\nearrow 0} ds$$

$$= W \cdot h$$

$$J(P) = \cancel{J(S_1)} + \cancel{J(S_2)} + J(S_3) + \cancel{J(S_4)} + \cancel{J(S_5)}$$

$$= W \cdot h$$

Review of fracture - different approaches.

Approach	Driving force	Fracture criterion
Griffith (Energy rel. rate)	$G = -\frac{\partial H}{\partial a}$	$G \geq \Gamma$
LEFM (stress. int. fact)	$\cdots \longrightarrow$	$K_I \geq K_{Ic}$; $K_{II} \geq K_{IIc}$; $K_{III} \geq K_{IIIc}$
J-integral	$J = \int_S (W dy - \underline{T} \cdot \frac{\partial u}{\partial x} ds)$	$J \geq J_c$

Cont'd.

$$\lim_{r \to 0} \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

$$\lim_{r \to 0} \sigma_{xy} = \frac{K_{II}}{\sqrt{2\pi r}}$$

$$\lim_{r \to 0} \sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}.$$