Functional Analysis

A Concept Map

Linear Algebra, Normed Spaces, Banach and Hilbert Spaces, Duality, Lebesgue and Sobolev Spaces, Traces, and FEM

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June 13, 2026

About this document

This is not a proof document. It is a **concept map**. I built it with the help of AI to understand functional analysis through simpler images. For every concept, ask:

- What object are we studying?
- What structure does it add?
- What theorem becomes possible because of that structure?
- What breaks if that structure is missing?

$$\text{Linear Algebra} \Rightarrow \text{Operators} \Rightarrow \text{Norm} \Rightarrow \text{Topology} \Rightarrow \text{Completeness}$$

$$\text{Hilbert Geometry} \Rightarrow \text{Duality} \Rightarrow L^p,\ H^1 \Rightarrow \text{Weak PDE/FEM}$$

Baby picture: The course is not random. It is building a machine: define the right space, prove the solution exists, then approximate it with FEM.

Contents

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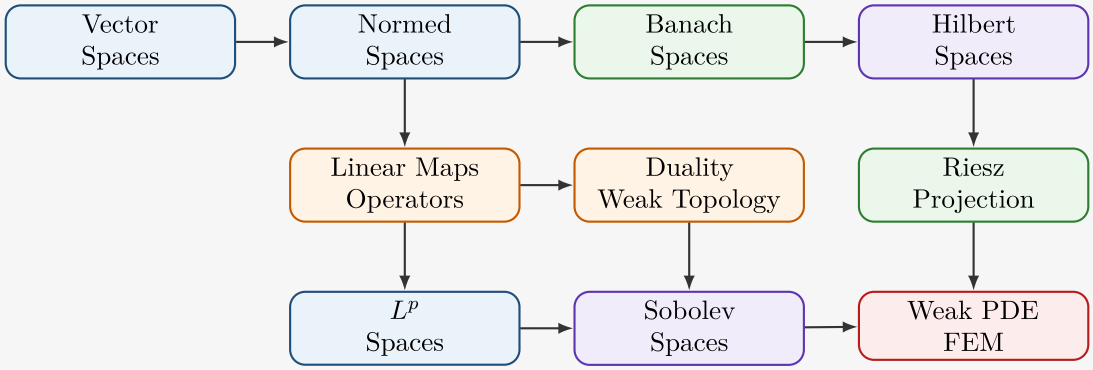
A. Representative Schematics

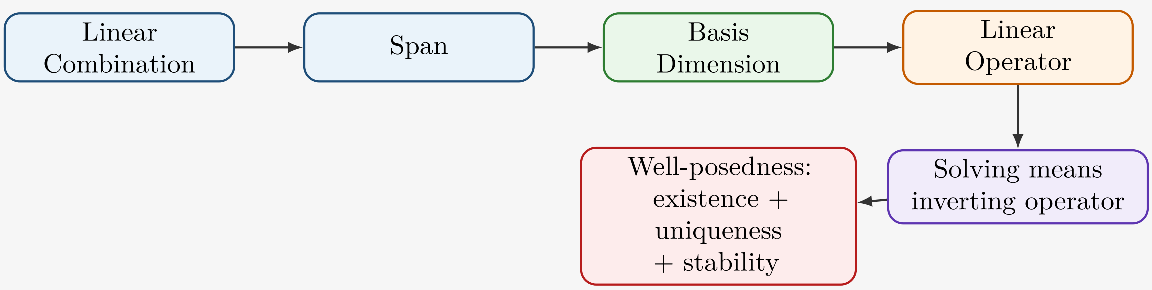
Schematic 1a. Master Roadmap of the Course



Meaning: The course starts with abstract vector spaces and ends with weak PDE formulations and finite element approximation.

Baby picture: First build the room, then add a ruler, then close the holes, then add geometry, then solve PDEs.

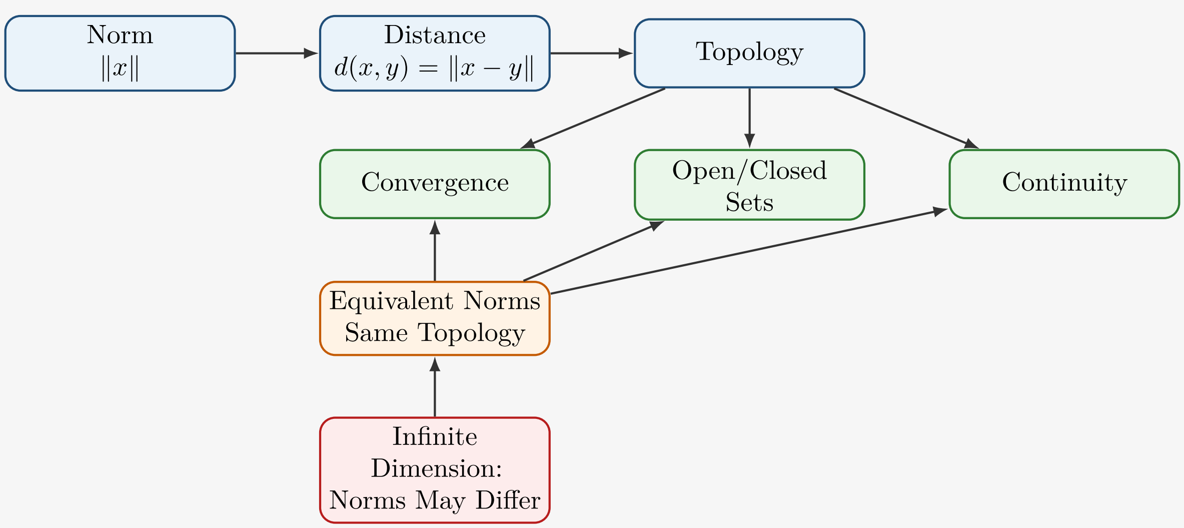
Schematic 1b. Linear Algebra to Operator Problems



PDE solving = operator inversion

Baby picture: Linear algebra tells us how finite-dimensional systems work. Functional analysis asks how the same ideas survive in infinite dimensions.

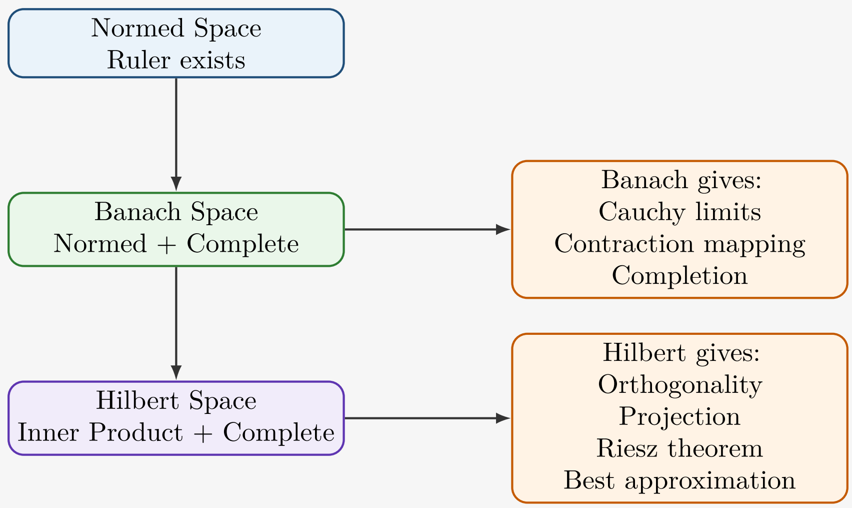
Schematic 2. Norms Define Topology



$$c\|x\|_a \le \|x\|_b \le C\|x\|_a \quad \Longrightarrow \quad \text{same convergence, open sets, closed sets, continuity.}$$

Baby picture: A norm is the ruler. Equivalent rulers distort lengths, but not enough to change what “close” means.

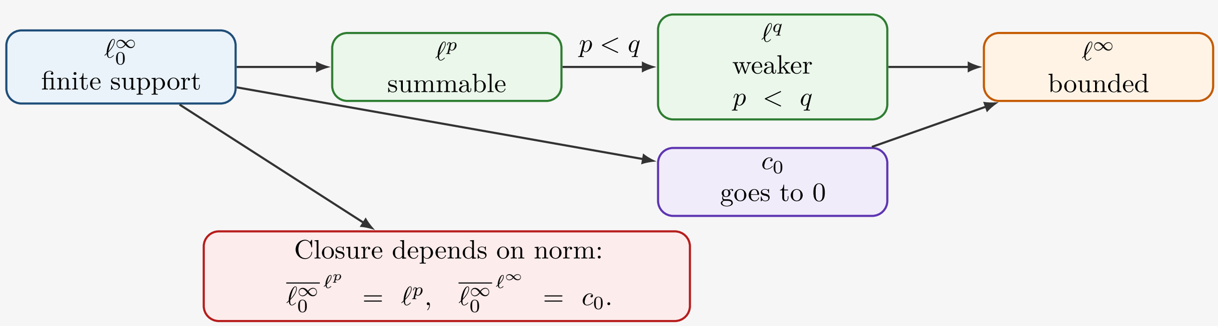
Schematic 3. Completeness Hierarchy



$$\text{Banach} = \text{complete normed space}, \qquad \text{Hilbert} = \text{complete inner-product space.}$$

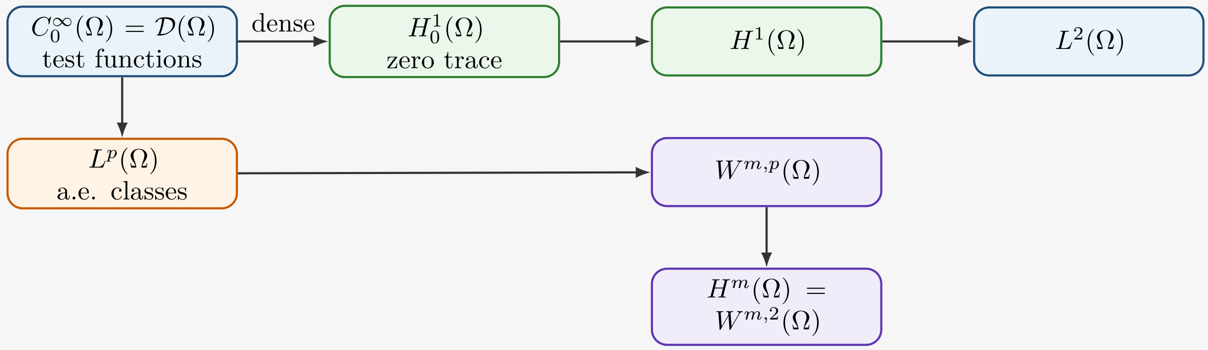
Baby picture: Banach means no holes. Hilbert means no holes plus angles.

Schematic 4. Sequence Space Relationships



Baby picture: In ℓ^p, chopping off the tail works. In ℓ^∞, the worst tail entry still matters.

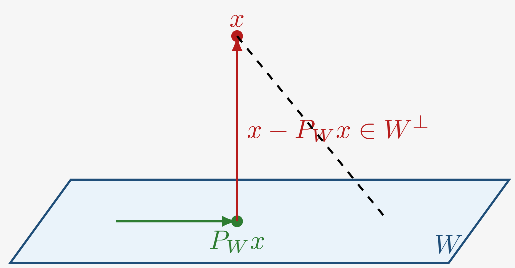
Schematic 5. Function Space and Sobolev Space Relationships



$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ |\alpha| \leq m\}.$$

Baby picture: Sobolev spaces are where rough functions can still have usable weak derivatives.

Schematic 6. Orthogonal Projection and Best Approximation

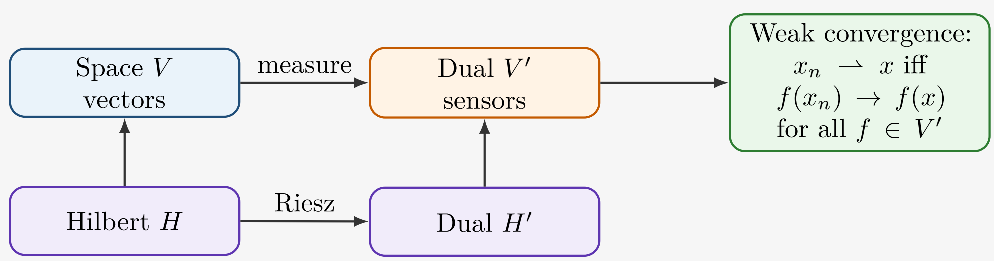


$$x = P_W x + (x - P_W x)$$
$$\|x - P_W x\| = \min_{w \in W} \|x - w\|$$

$$H = W \oplus W^\perp \qquad \text{if } H \text{ is Hilbert and } W \text{ is closed.}$$

Baby picture: Projection is dropping a perpendicular shadow onto the subspace.

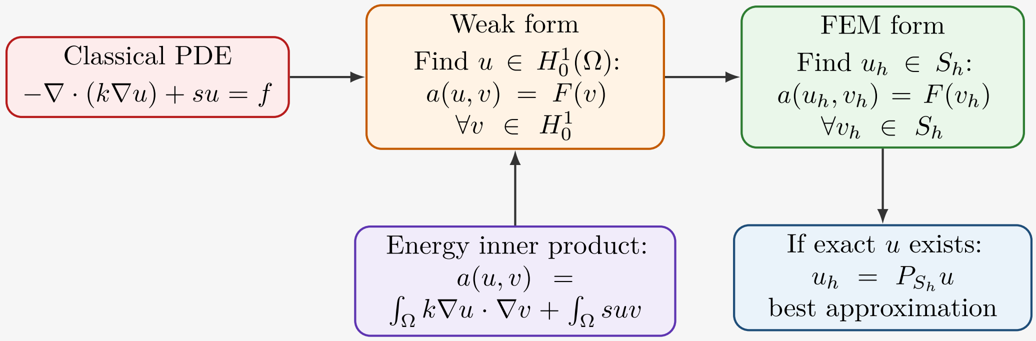
Schematic 7. Duality, Riesz, and Weak Convergence



$$f(y) = (y, x_f), \qquad \|f\|_{H'} = \|x_f\|_H.$$

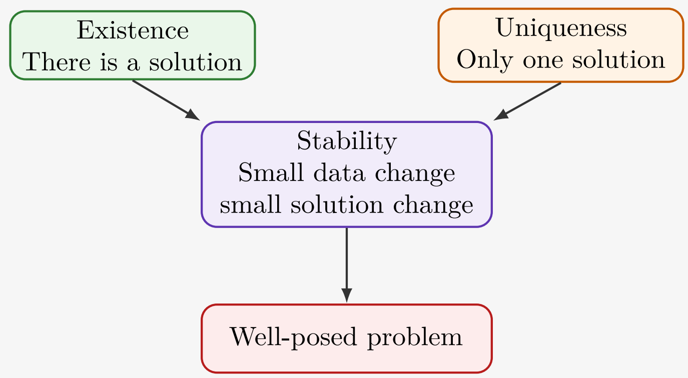
Baby picture: A functional is a sensor. Weak convergence means every sensor reading converges.

Schematic 8a. PDE to Weak Form to FEM



Baby picture: FEM is the best shadow of the true solution inside a finite-dimensional trial space.

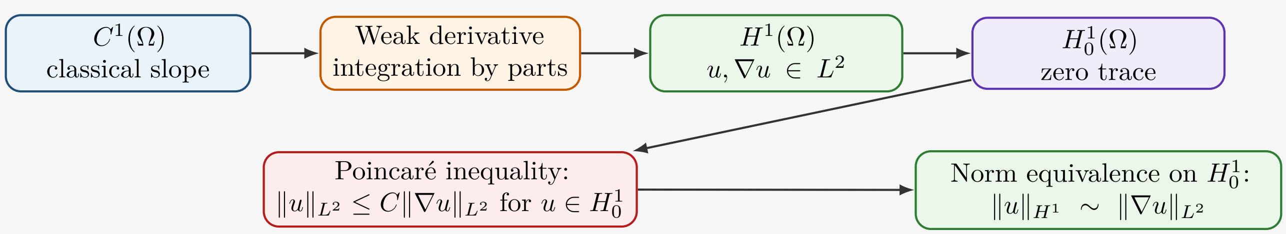
Schematic 8b. Well-Posedness Triangle



well-posed = existence + uniqueness + continuous dependence

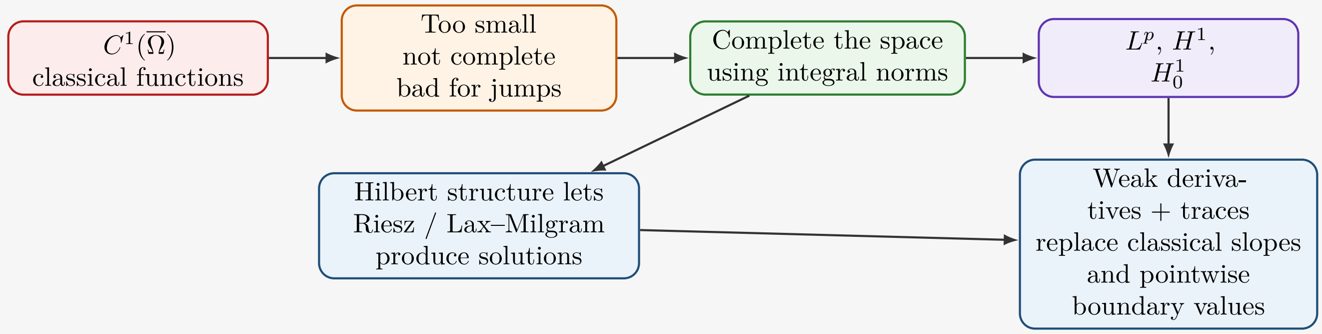
Baby picture: A problem is useful only if the solution exists, is unique, and does not explode when the input data wiggles.

Schematic 9. HW5 Sobolev Energy-Space Bridge



Baby picture: HW5 sharpens the PDE space idea: functions may not be classically smooth, but if their weak slope has finite energy and the boundary is tied down, the slope controls the whole function.

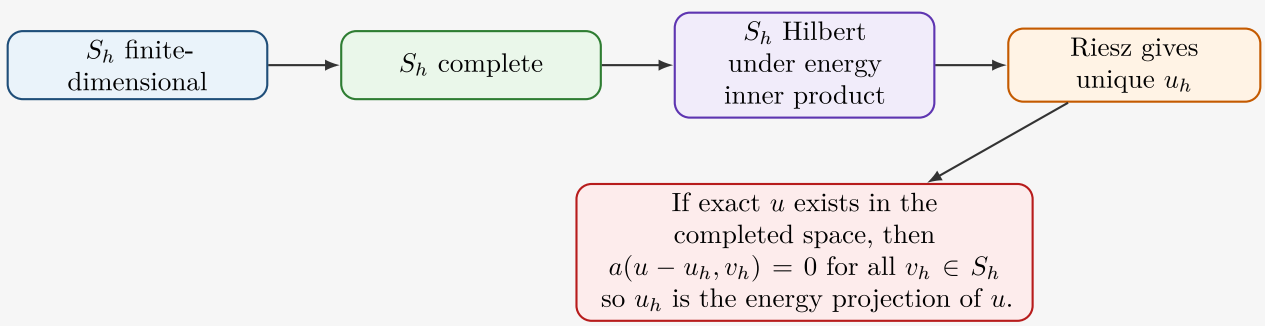
Schematic 10. Why Lebesgue and Sobolev Spaces Enter



C^1 is often the wrong room; Sobolev spaces are the completed energy rooms.

Baby picture: Lebesgue measure tells us what "almost everywhere" means. Sobolev spaces then let PDE solutions be rough but still controllable.

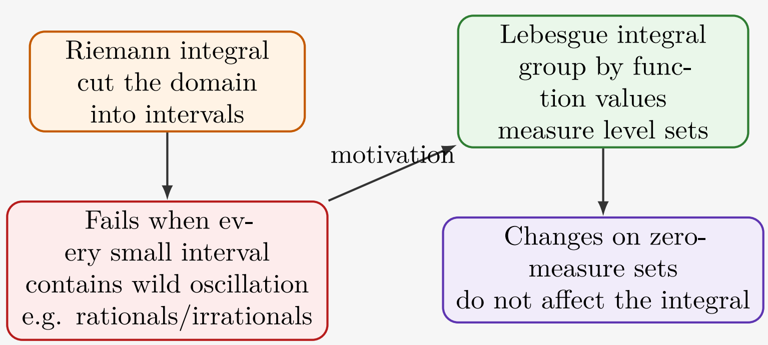
Schematic 11. FEM Existence and Best Approximation Logic



FEM existence is easier than infinite-dimensional existence because S_h is finite-dimensional.

Baby picture: Even if the full classical space has holes, the finite-dimensional FEM room has no holes.

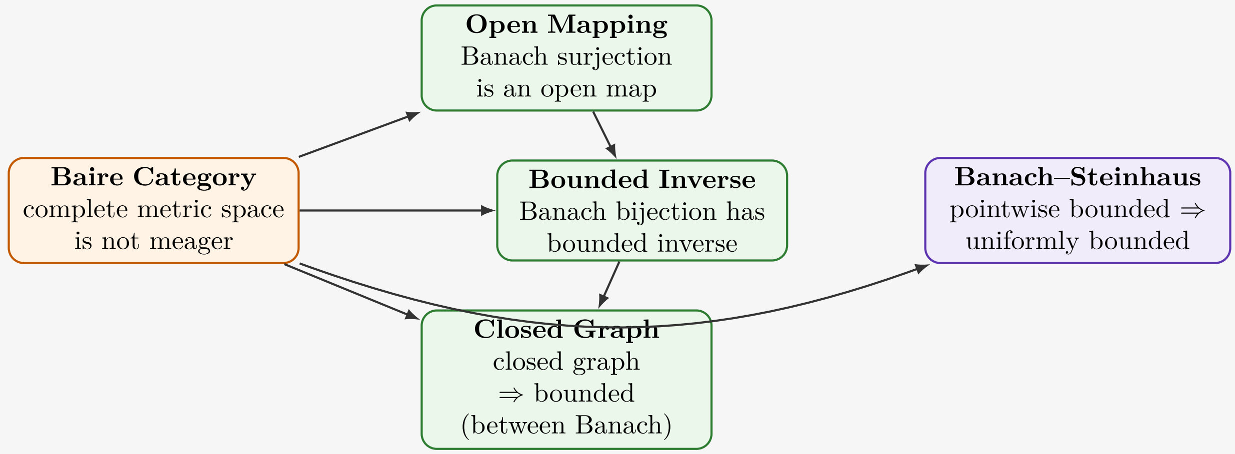
Schematic 12. Riemann vs Lebesgue Integration



$$\chi_{\mathbb{Q}} \text{ is not Riemann integrable on } [0,1], \qquad \int_0^1 \chi_{\mathbb{Q}}\,dx = 0 \text{ Lebesgue.}$$

Baby picture: Riemann asks: what happens on each little interval? Lebesgue asks: how much mass takes each value?

Schematic 13. The “Big 4” of Functional Analysis

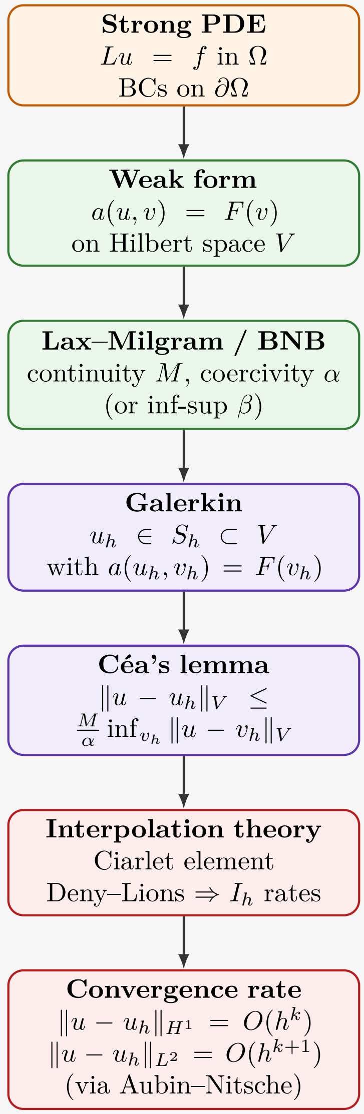


Baire ⇒ Open Mapping ⇔ Bounded Inverse ⇔ Closed Graph.

Baire ⇒ Uniform Boundedness Principle.

Baby picture: All four “Big” theorems descend from Baire category. They are the structural backbone that makes Banach-space analysis work.

Schematic 14. The Variational + FEM Ladder



Reading. The PDE descends to a weak form on a Hilbert space; coercivity (Lax–Milgram) or inf-sup (BNB) buys existence and stability; the discrete Galerkin solution inherits quasi-optimality from Céa; concrete convergence rates come from interpolation theory and (for L^2) Aubin–Nitsche duality.

Baby picture: Existence climbs up the ladder; convergence rates come back down through interpolation.

B. Core Concepts

Part 0. Linear Algebra Foundations

0a. Linear Combination

Given vectors $v_1, \dots, v_n$, a linear combination is

$$\alpha_1 v_1 + \cdots + \alpha_n v_n.$$

Baby picture: A linear combination is a recipe: take some amount of each vector and mix them.

0b. Span

The span of vectors $v_1, \dots, v_n$ is

$$\operatorname{span}\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^{n} \alpha_i v_i : \alpha_i \in \mathbb{R} \right\}.$$

Baby picture: Span is everything you can build from your vector ingredients.

0c. Linear Independence

Vectors $v_1, \dots, v_n$ are linearly independent if

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \quad \Rightarrow \quad \alpha_1 = \cdots = \alpha_n = 0.$$

Baby picture: Linear independence means no vector is secretly made from the others. No redundancy.

0d. Basis and Dimension

A basis is a linearly independent set that spans the whole space.

$$\text{basis} = \text{independent construction kit for the space.}$$

The number of basis vectors is the dimension.
Baby picture: A basis is the minimal LEGO kit that can build every vector in the space.

0e. Affine Combination

An affine combination has the form

$$\sum_{i=1}^{n} \lambda_i x_i, \qquad \sum_{i=1}^{n} \lambda_i = 1.$$

Baby picture: A linear combination combines directions. An affine combination averages points.

0f. Convex Combination and Convex Set

A convex combination has

$$\sum_{i=1}^{n} \lambda_i x_i, \qquad \lambda_i \geq 0, \qquad \sum_{i=1}^{n} \lambda_i = 1.$$

A set C is convex if

$$x, y \in C, \quad 0 \leq \lambda \leq 1 \quad \Rightarrow \quad \lambda x + (1-\lambda) y \in C.$$

Baby picture: Convex means: if two points are inside, the straight line between them stays inside.

0g. Direct Sum

We write

$$V = X \oplus Y$$

if every $v \in V$ has a unique decomposition

$$v = x + y, \qquad x \in X, \quad y \in Y.$$

Equivalent conditions:

$$X + Y = V, \qquad X \cap Y = \{0\}.$$

Baby picture: Direct sum means every vector has one clean split into an X-part and a Y-part.

Part I. Normed-Space Foundations

1. Vector Space

A vector space is a set where addition and scalar multiplication make sense.

$$x, y \in V,\ \alpha, \beta \in \mathbb{R} \quad \Rightarrow \quad \alpha x + \beta y \in V.$$

Baby picture: A vector space is a playground where linear combinations are legal.

2. Seminorm vs Norm

A seminorm satisfies positivity, homogeneity, and triangle inequality, but may have

$$p(x) = 0 \quad \text{for some } x \neq 0.$$

A norm additionally requires:

$$\|x\| = 0 \iff x = 0.$$

Baby picture: A seminorm may think two different objects have zero distance. A norm only gives zero size to the actual zero vector.

3. Norm

A norm measures vector size:

$$\|x\| \geq 0, \qquad \|x\| = 0 \iff x = 0,$$

$$\|\alpha x\| = |\alpha|\|x\|, \qquad \|x + y\| \leq \|x\| + \|y\|.$$

It induces the distance:

$$d(x, y) = \|x - y\|.$$

Baby picture: Norm = ruler. Once you choose the ruler, you define what closeness means.

4. Open and Closed Sets

A set A is open if every point has a small ball inside:

$$x \in A \Rightarrow \exists r > 0 \text{ such that } B_r(x) \subset A.$$

A set C is closed if its complement is open. Equivalently, it contains limits of convergent sequences:

$$x_n \in C, \quad x_n \to x \quad \Rightarrow \quad x \in C.$$

Baby picture: Open set = you can wiggle. Closed set = limits cannot escape.

5. Convergence

$$x_n \to x \iff \|x_n - x\| \to 0.$$

Baby picture: The sequence is landing on the point x.

6. Cauchy Sequence

$$\forall \varepsilon > 0,\ \exists N, \quad m, n \geq N \Rightarrow \|x_n - x_m\| < \varepsilon.$$

A Cauchy sequence clusters internally, even before we know whether the limit exists.
Baby picture: The crowd huddles tighter and tighter. Completeness tells you whether the huddle lands inside the room.

7. Completeness and Banach Spaces

A normed space is complete if every Cauchy sequence converges inside the space.

$$\text{Banach space} = \text{complete normed vector space.}$$

Key point: Completeness is the first serious existence mechanism.
Baby picture: Banach = normed space with no holes.

8. Closed Subspace of a Banach Space

If V is Banach and $W \subset V$ is closed, then W is Banach.

$$V \text{ complete}, \quad W \text{ closed} \quad \Rightarrow \quad W \text{ complete.}$$

Baby picture: A sealed room inside a no-hole building also has no holes.

9. Completion

Every normed space is either already complete or can be embedded densely into a complete space.

$$V \subset \overline{V}, \qquad \overline{V} \text{ Banach.}$$

Construction (Cauchy-class completion):

- Let $\mathcal{C}(V)$ be the set of all Cauchy sequences in V.
- Define $(x_n) \sim (y_n)$ if $\|x_n - y_n\| \to 0$.
- Set $\overline{V} := \mathcal{C}(V)/\!\sim$ with $\|[x_n]\| := \lim_n \|x_n\|$.
- The map $x \mapsto [(x, x, \dots)]$ embeds V isometrically as a dense subspace of the Banach space $\overline{V}$.

Key point: The completion is unique up to isometric isomorphism that fixes V.
Baby picture: Completion fills the holes. Rationals become reals; smooth functions become Sobolev spaces.

10. Quotient Space

If $W \subset V$ is a closed subspace, the quotient V/W identifies vectors that differ by an element of W.

$$[x] = x + W.$$

The norm is

$$\|[x]\|_{V/W} = \inf_{w \in W} \|x + w\|_V.$$

Baby picture: A quotient space says: if two vectors differ only by something we decide to ignore, treat them as the same.

Part II. Norm Equivalence, Density, and Compactness

11. Equivalent Norms

Two norms are equivalent if

$$c\|x\|_a \le \|x\|_b \le C\|x\|_a.$$

Equivalent norms produce the same:

- convergence,
- Cauchy sequences,
- open and closed sets,
- continuity.

Baby picture: Different rulers, same world.

12. Finite vs Infinite Dimension

In finite-dimensional spaces:

all norms are equivalent.

In infinite-dimensional spaces:

norms may not be equivalent.

Consequences:

- closure depends on norm,
- convergence depends on norm,
- bounded does not imply compact.

Baby picture: Finite dimension is a safe city. Infinite dimension is an ocean.

13. Closure

The closure $\overline{A}$ is the set of all limits of sequences in A.

$$x \in \overline{A} \iff \exists x_n \in A,\ x_n \to x.$$

Baby picture: Closure means everything you can approach using points from the set.

14. Density

A is dense in V if

$$\overline{A} = V.$$

Baby picture: Dense means you can approximate every target using elements of A.

15. Compactness

A set K is compact if every sequence in K has a convergent subsequence.

$$x_n \in K \quad \Rightarrow \quad \exists x_{n_k} \to x \in K.$$

Baby picture: Compactness means no infinite sequence can wander forever without some subsequence settling down.

16. Compactness in Infinite Dimension

In $\mathbb{R}^n$, closed and bounded implies compact.
In infinite-dimensional spaces, this is false.

$$\{e_n\} \subset \ell^2$$

is bounded, but has no strongly convergent subsequence.
Baby picture: In infinite dimension, bounded sets can still escape by moving into new directions forever.

16a. Riesz's Theorem: Compact Unit Ball Detects Finite Dimension

For a normed vector space V:

$$\overline{B_1(0)} \text{ compact} \iff \dim V < \infty.$$

Equivalent characterizations:

- Every bounded sequence in V has a convergent subsequence.
- Closed and bounded sets are compact in V.

Key point: This is the cleanest topological test that separates finite-dimensional spaces from genuinely infinite-dimensional ones.
Baby picture: Finite dimension is exactly the world where a closed bounded ball cannot "leak" into infinitely many new directions.

16b. Compact Sets Beyond Finite Dimension

Even in infinite-dimensional spaces, useful compact sets exist; they just have to enforce decay of tails or smoothness. In $\ell^2(\mathbb{R})$:

$$K = \left\{ x \in \ell^2 : \sum_{n=1}^{\infty} n^2 x_n^2 \le 1 \right\}$$

is compact.
Baby picture: Compactness in infinite dimensions costs you something — in this example a uniform weighted-tail decay.

17. Fixed Point Theorems

Contraction mapping theorem: A contraction on a closed subset of a Banach space has a unique fixed point.

$$\|T(x) - T(y)\| \le c\|x - y\|, \qquad c < 1.$$

Brouwer fixed point theorem: A continuous map from a compact convex set in $\mathbb{R}^n$ into itself has at least one fixed point.
Schauder fixed point theorem (infinite-dimensional Brouwer): A continuous map from a compact convex subset of a Banach space into itself has at least one fixed point. A useful precompact-image variant exists for self-maps of closed bounded convex sets.
Baby picture: Contraction gives existence and uniqueness. Brouwer/Schauder give existence from compactness, usually not uniqueness.

Part III. Sequence Spaces

18. ℓ^p, ℓ^∞, c_0, and ℓ_0^∞

$$\ell^p = \left\{ x = (x_i) : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}, \qquad 1 \le p < \infty.$$

$$\ell^\infty = \{x = (x_i) : \sup_i |x_i| < \infty\}.$$

$$c_0 = \{x = (x_i) : x_i \to 0\}.$$

$$\ell_0^\infty = \{x = (x_i) : x_i = 0 \text{ except finitely many } i\}.$$

Baby picture: ℓ^p means summable tail. ℓ^∞ means no blow-up. c_0 means fading to zero. ℓ_0^∞ means finite support.

19. Inclusion of Sequence Spaces

For $1 \le p < q \le \infty$,

$$\ell^p \subset \ell^q.$$

Reason:

$$x \in \ell^p \Rightarrow x_i \to 0.$$

For small entries,

$$|x_i|^q \le |x_i|^p.$$

Baby picture: If a sequence passes the strict test, it passes the weaker test.

20. Closure of ℓ_0^∞

For $1 \le p < \infty$,

$$\overline{\ell_0^\infty}^{\ell^p} = \ell^p.$$

But in ℓ^∞,

$$\overline{\ell_0^\infty}^{\ell^\infty} = c_0.$$

Trap: Same set, different norm, different closure.

Part IV. Inner Product and Hilbert Geometry

21. Inner Product

An inner product (x, y) gives a norm:

$$\|x\| = \sqrt{(x, x)}.$$

It allows:

- length,
- angle,
- orthogonality,
- projection.

Baby picture: Norm gives a ruler. Inner product gives a ruler plus a protractor.

22. Cauchy–Schwarz

$$|(x, y)| \le \|x\| \, \|y\|.$$

Baby picture: Dot product cannot exceed length times length.

23. Parallelogram Law and Polarization

A norm comes from an inner product iff it satisfies:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

For real inner product spaces:

$$(x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right).$$

Baby picture: The parallelogram law detects whether your ruler secretly came from angles.

24. Orthogonal Complement

$$W^\perp = \{x \in H : (x, w) = 0 \ \forall w \in W\}.$$

Useful properties:

$$W^\perp = \overline{W}^\perp, \qquad W \subset W^{\perp\perp}.$$

If H is Hilbert:

$$W^{\perp\perp} = \overline{W}.$$

Baby picture: The orthogonal complement is everything invisible to W under the inner product.

24a. Concrete Orthogonal Complement Example in ℓ^2

Let

$$W = \{x \in \ell^2 : x = (0, 0, x_3, x_4, \dots)\}.$$

Then

$$W^\perp = \{z \in \ell^2 : z = (z_1, z_2, 0, 0, \dots)\}.$$

Reason: if $z \perp W$, then in particular $z \perp e_i$ for every $i \geq 3$, hence $z_i = 0$ for $i \geq 3$. Conversely, any sequence with only the first two entries possibly nonzero is orthogonal to every element of W.
Baby picture: If W lives only in coordinates $3, 4, 5, \dots$, then $W^\perp$ lives only in coordinates $1, 2$.

25. Hilbert Space

$$\text{Hilbert space} = \text{complete inner-product space.}$$

Hilbert spaces are the closest infinite-dimensional spaces to Euclidean geometry.
Baby picture: Hilbert = perfect geometry room: no holes, with angles.

26. Orthogonal Projection Theorem

If H is Hilbert and $W \subset H$ is closed, then every $x \in H$ has a unique projection $P_W x \in W$:

$$x = P_W x + (x - P_W x), \qquad x - P_W x \in W^\perp.$$

Baby picture: Drop a perpendicular shadow onto W.

27. Best Approximation

The projection is the closest point:

$$\|x - P_W x\| = \min_{w \in W} \|x - w\|.$$

Baby picture: Projection is the least-wrong approximation.

28. Orthogonal Decomposition

If H is Hilbert and W is closed:

$$H = W \oplus W^\perp.$$

This means every $x \in H$ has a unique split:

$$x = w + z, \qquad w \in W, \ z \in W^\perp.$$

Baby picture: Every vector splits into floor shadow plus vertical error.

29. Failure Without Completeness

If the space is not complete, projection may fail.
Example idea:

$$W^{\perp} = \{0\} \quad \text{but} \quad W \neq V.$$

Then:

$$V \neq W \oplus W^{\perp}.$$

Baby picture: If the room has holes, the shadow may fall into a missing point.

29a. Existence of an Orthonormal Basis

Every Hilbert space H admits an orthonormal basis: a maximal orthonormal set $A \subset H$ such that

$$\overline{\operatorname{span}(A)} = H.$$

Proof idea: apply Zorn's lemma to chains of orthonormal sets ordered by inclusion.
Key point: The basis can be uncountable in general, but if H is separable, the basis is at most countable.
Baby picture: Every Hilbert space has a coordinate system, even when the space is infinite-dimensional.

29b. Separability and Countable Orthonormal Bases

A Hilbert space H is separable (has a countable dense subset) if and only if it admits a countable orthonormal basis.
Baby picture: Separable means "not too big to coordinatize with $\mathbb{N}$."

29c. Separable Hilbert Spaces Are All $\ell^2(\mathbb{K})$

Every separable Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{K})$:

$$H \cong \ell^2(\mathbb{K}), \qquad x \longmapsto \big((x, e_i)\big)_{i\in\mathbb{N}}.$$

Inverse:

$$(c_i) \longmapsto \sum_{i=1}^{\infty} c_i e_i.$$

Key point: Up to isometric isomorphism, there is exactly one separable infinite-dimensional Hilbert space.
Baby picture: The full geometry of a separable Hilbert space is captured by square-summable coordinate sequences.

Part V. Orthonormal Systems and Approximation

30. Orthonormal System

$$(e_i, e_j) = \delta_{ij}.$$

Baby picture: Like perpendicular coordinate axes with unit length.

31. Fourier-Type Approximation

For $S_n = \operatorname{span}\{e_1, \ldots, e_n\}$, the best approximation is:

$$f_n = \sum_{i=1}^{n} (f, e_i) e_i.$$

Baby picture: Each coefficient measures how much f points in the e_i direction.

32. Bessel's Inequality

For any orthonormal system $\{e_i\}_{i\in I} \subset H$ and any $x \in H$,

$$\sum_{i\in I} |(x, e_i)|^2 \le \|x\|^2.$$

Consequence: the Fourier-type partial sums

$$S_J(x) = \sum_{i\in J} (x, e_i)e_i$$

converge for every countable $J \subset I$.
Baby picture: Coordinates with respect to an orthonormal system cannot store more energy than the original vector.

32a. Parseval's Identity and Completeness Equivalences

For an orthonormal system $A \subset H$ in a Hilbert space, the following are equivalent:

1. A is an orthonormal basis ($\overline{\text{span}(A)} = H$).
2. **Fourier expansion** holds for every $x \in H$:
$$x = \sum_{e\in A} (x, e)e.$$
3. **Parseval's identity:**
$$(x, y) = \sum_{e\in A} (x, e)\overline{(y, e)}.$$
4. **Bessel equality:**
$$\|x\|^2 = \sum_{e\in A} |(x, e)|^2.$$

Key point: Parseval upgrades Bessel from inequality to equality exactly when the orthonormal system is a basis.
Baby picture: A complete basis accounts for all of the energy and all of the geometry.

Part VI. Linear Operators, Duality, and Weak Topology

33. Linear Operator

A linear operator is a map

$$T : V \to W$$

such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Baby picture: A linear operator is the infinite-dimensional version of a matrix.

34. Kernel and Image

The kernel is

$$\ker(T) = \{x \in V : Tx = 0\}.$$

The image/range is

$$\text{Im}(T) = T(V) = \{Tx : x \in V\}.$$

Baby picture: The kernel is what the operator kills. The image is what the operator can produce.

35. Injective, Surjective, Bijective

$$T \text{ injective} \iff Tx = Ty \Rightarrow x = y.$$

For linear maps:

$$T \text{ injective} \iff \ker(T) = \{0\}.$$

$$T \text{ surjective} \iff \operatorname{Im}(T) = W.$$

$$T \text{ bijective} \iff \text{injective and surjective.}$$

Baby picture: Injective = no collapse. Surjective = hits every target. Bijective = every output has exactly one input.

36. Operator Inversion and PDE Solving

For an operator equation

$$Au = f,$$

solving means finding

$$u = A^{-1} f.$$

Existence means $f \in \operatorname{Im}(A)$. Uniqueness means $\ker(A) = \{0\}$. Stability means A^{-1} is continuous.
Baby picture: A PDE is an operator equation. Solving the PDE means inverting the operator.

37. Well-Posedness

A problem is well-posed if it has:

1. **Existence:** a solution exists.
2. **Uniqueness:** the solution is unique.
3. **Continuous dependence:** small changes in data produce small changes in solution.

For

$$Au = f,$$

this means A should be invertible and A^{-1} should be continuous.
Baby picture: A good problem has a solution, only one solution, and does not panic when the input wiggles.

38. Bounded Linear Map

A linear map $T : V \to W$ is bounded if:

$$\|Tx\|_W \le M \|x\|_V.$$

For linear maps:

$$\boxed{\text{bounded} \iff \text{continuous}}$$

Baby picture: Bounded means the operator cannot stretch inputs uncontrollably.

39. Operator Norm

$$\|T\|_{L(V,W)} = \sup_{x \neq 0} \frac{\|Tx\|_W}{\|x\|_V} = \sup_{\|x\|_V = 1} \|Tx\|_W.$$

Baby picture: Operator norm measures the maximum stretch of the operator.

40. Dual Space

The topological dual is:

$$V' = L(V, \mathbb{R}).$$

It consists of all continuous linear functionals on V.
Baby picture: The dual space is the collection of all continuous sensors that measure vectors.

41. Hahn–Banach (Extension Form)

Real version. Let V be a real vector space, $p : V \to \mathbb{R}$ a sublinear functional ($p(\alpha x) = \alpha p(x)$ for $\alpha \geq 0$ and $p(x + y) \leq p(x) + p(y)$), and f a linear functional on a subspace $W \subset V$ with $f \leq p$ on W. Then f extends to a linear functional $\tilde{f} : V \to \mathbb{R}$ with $\tilde{f} \leq p$ everywhere.
Complex version. Replace p by a seminorm and require $|f(x)| \leq p(x)$ on W. Then the extension $\tilde{f}$ satisfies $|\tilde{f}(x)| \leq p(x)$ on all of V.
Norm-extension corollary (Banach space version). If $W \subset V$ is a subspace and $f \in W'$, then f has an extension $\tilde{f} \in V'$ with $\|\tilde{f}\|_{V'} = \|f\|_{W'}$.
Separation corollary. Disjoint convex sets in a normed space (one with nonempty interior) can be separated by a continuous linear functional.
Baby picture: A sensor defined on a smaller world can be extended to the larger world without becoming more violent. This is why normed and Banach spaces have rich duals.

42. Riesz Representation Theorem

If H is Hilbert, then every $f \in H'$ has a unique vector $x_f \in H$ such that:

$$f(y) = (y, x_f) \qquad \forall y \in H.$$

Also:

$$\|f\|_{H'} = \|x_f\|_H.$$

Baby picture: Every continuous sensor in a Hilbert space is just an inner product with a hidden vector.

43. Riesz Proof Geometry

For a nonzero functional f,

$$\ker(f)$$

is a hyperplane.
In a Hilbert space,

$$H = \ker(f) \oplus \ker(f)^\perp.$$

Moreover,

$$\dim \ker(f)^\perp = 1.$$

Baby picture: A nonzero functional kills a whole hyperplane and only sees one perpendicular direction. That direction is the Riesz representative.

44. Kernel Geometry Under Riesz

For $f(y) = (y, x_f)$,

$$\ker(f) = \{y : (y, x_f) = 0\} = x_f^\perp.$$

Thus:

$$\ker(f)^\perp = \operatorname{span}\{x_f\}.$$

Baby picture: The functional detects one direction. Everything perpendicular to that direction is invisible.

45. Strong vs Weak Convergence

Strong convergence:

$$x_n \to x \quad \iff \quad \|x_n - x\| \to 0.$$

Weak convergence:

$$x_n \rightharpoonup x \quad \iff \quad f(x_n) \to f(x) \quad \forall f \in V'.$$

Strong implies weak:

$$x_n \to x \Rightarrow x_n \rightharpoonup x.$$

Baby picture: Strong convergence means the vector gets close. Weak convergence means every sensor reading gets close.

46. Weak Does Not Mean Strong

In ℓ^2, the canonical basis e_n satisfies:

$$e_n \rightharpoonup 0,$$

but

$$\|e_n - 0\| = 1.$$

So e_n does not converge strongly to 0.
Trap: Weak convergence is weaker than norm convergence.

47. Weak-Strong Compatibility Pattern

A useful pattern:
If

$$x_n \rightharpoonup x$$

and $T : V \to W$ is bounded linear, then

$$Tx_n \rightharpoonup Tx.$$

If additionally

$$Tx_n \to y \quad \text{strongly},$$

then necessarily

$$y = Tx.$$

Baby picture: Weak limits and strong limits cannot disagree when a bounded linear operator connects them correctly.

47a. Open Mapping Theorem

Let V and W be Banach spaces and $T : V \to W$ a surjective bounded linear operator. Then T is an open map: the image of every open set in V is open in W.
Baby picture: A continuous linear surjection between Banach spaces cannot squish open sets into something thinner.

47b. Bounded Inverse Theorem

If V and W are Banach and $T : V \to W$ is a bounded linear bijection, then $T^{-1} : W \to V$ is bounded.

$$T \text{ bounded linear bijection between Banach spaces} \implies T^{-1} \in L(W, V).$$

Key point: This is the existence-uniqueness-stability glue for well-posedness.
Baby picture: A continuous bijection between complete spaces automatically has a continuous inverse.

47c. Closed Graph Theorem

Let V and W be Banach spaces and $T : V \to W$ linear. Then

$$T \text{ bounded} \iff \text{graph}(T) = \{(x, Tx) : x \in V\} \text{ is closed in } V \times W.$$

Baby picture: If the graph does not leak, the operator is continuous — you do not need to test continuity directly.

47d. Uniform Boundedness Principle (Banach–Steinhaus)

Let V be Banach, W normed, and $\{T_\alpha\}_{\alpha \in A} \subset L(V, W)$ a family of bounded linear operators. If for every $x \in V$

$$\sup_\alpha \|T_\alpha x\| < \infty,$$

then

$$\sup_\alpha \|T_\alpha\| < \infty.$$

Pointwise boundedness $\Rightarrow$ **uniform boundedness**, on Banach spaces.
Key point: Together with Open Mapping, Bounded Inverse, and Closed Graph, this is one of the "Big Four" theorems of functional analysis, all consequences of Baire category.
Baby picture: If every test vector sees bounded operator outputs, then the whole family is uniformly bounded.

47e. Bidual, Canonical Embedding, and Reflexive Spaces

For a normed space V, define the canonical embedding

$$J : V \to V'', \qquad (Jx)(f) = f(x).$$

J is always an isometric embedding.
Definition. V is **reflexive** if J is surjective, i.e. $V \cong V''$ via J.
Examples:

- Every Hilbert space is reflexive (via two Riesz applications).
- $L^p(\Omega)$ is reflexive for $1 < p < \infty$.
- L^1, L^∞, c_0 are not reflexive.

Baby picture: Reflexive means the space is its own "sensor space of sensors" — no new vectors appear when you take duals twice.

47f. Weak-* Topology

On the dual space V', the **weak-*** topology is the coarsest topology making each evaluation map

$$\hat{x} : f \mapsto f(x), \quad x \in V,$$

continuous. Equivalently,

$$f_n \xrightarrow{w*} f \iff f_n(x) \to f(x) \quad \forall x \in V.$$

Key point: Weak-* compactness is stronger than weak compactness on a dual, and is the key tool behind variational existence proofs.
Baby picture: Weak-* convergence of sensors means every fixed sample reading converges.

47g. Banach–Alaoglu Theorem

The closed unit ball of V' is compact in the weak-* topology:

$$\overline{B_1^{V'}(0)} \text{ is weak-* compact.}$$

For separable V, weak-* compactness on bounded sets is also sequential.
Key point: This is the abstract existence engine: any bounded family of functionals has a weak-* convergent subnet

(subsequence for separable V).
Baby picture: Even though norm-compact balls are tiny in infinite dimensions, weak-* compact balls are easy to come by.

47h. Weak Compactness in Hilbert Spaces

In a Hilbert space H, every bounded sequence has a weakly convergent subsequence:

$$\sup_n \|x_n\| < \infty \implies \exists x_{n_k} \rightharpoonup x \in H.$$

This follows from Banach–Alaoglu in the reflexive case combined with Riesz representation.
Key point: In Hilbert spaces, weak compactness gives a clean "boundedness ⇒ weakly convergent subsequence" principle.
Baby picture: Weak compactness gives an existence trick: extract a weak limit from a bounded minimizing sequence.

Part VII. Lebesgue and L^p Spaces

48. Zero Measure

A set has zero measure if it can be covered by countably many tiny balls whose total volume can be made arbitrarily small.
More explicitly, $X \subset \mathbb{R}^n$ has zero measure if for every $\varepsilon > 0$, there are balls $B_{r_i}(x_i)$ such that

$$X \subset \bigcup_{i\in\mathbb{N}} B_{r_i}(x_i), \qquad \sum_{i\in\mathbb{N}} r_i^n < \varepsilon.$$

Examples:

$$\mathbb{Q} \subset \mathbb{R} \quad \text{has zero measure,} \qquad \{(x,y) : x = 0\} \subset \mathbb{R}^2 \quad \text{has zero measure.}$$

Baby picture: Zero-measure sets are mathematical dust: possibly infinitely many points, but no volume.

49. Almost Everywhere

A statement holds almost everywhere, abbreviated a.e., if it fails only on a zero-measure set.

$$f = g \text{ a.e.} \quad \iff \quad |\{x : f(x) \neq g(x)\}| = 0.$$

A.e. limits are only unique up to a.e. equality. So two different pointwise functions may represent the same L^p object.
Baby picture: In L^p, changing a function on dust does not count.

50. Riemann vs Lebesgue Integral

Riemann integration cuts the domain into intervals and checks upper/lower oscillation. Lebesgue integration groups points by function value and measures how much of the domain takes each value.
For the Dirichlet function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

Riemann integration fails because every interval contains both rationals and irrationals. But Lebesgue integration gives

$$\int_0^1 \chi_{\mathbb{Q}}\, dx = 0,$$

because $\mathbb{Q} \cap [0,1]$ has measure zero.
Baby picture: Riemann gets confused by dense dust. Lebesgue weighs the dust and sees it has zero mass.

51. Measurable and Simple Functions

A function $f : \mathbb{R}^n \to [-\infty, +\infty]$ is measurable if

$$\{x : f(x) > a\} \in \mathcal{M} \qquad \forall a \in \mathbb{R}.$$

A simple function takes finitely many values:

$$s(x) = \sum_{j=1}^{m} a_j \chi_{A_j}(x),$$

where the A_j are measurable sets.
The Lebesgue integral is first defined for simple functions:

$$\int_A s(x)\,dx = \sum_{j=1}^{m} a_j |A_j|,$$

then extended to nonnegative measurable functions, then to general real-valued functions using $f = f^+ - f^-$.
Baby picture: Simple functions are step functions with measurable steps. Lebesgue integration builds complicated functions by approximating them with these steps.

52. $L^p(\Omega)$

For $1 \le p < \infty$:

$$L^p(\Omega) = \left\{ f : \int_\Omega |f|^p < \infty \right\} / \text{a.e. equality}.$$

Norm:

$$\|f\|_{L^p} = \left(\int_\Omega |f|^p \right)^{1/p}.$$

Why quotient by a.e. equality? Because

$$\int_\Omega |f|^p = 0 \quad \Longleftrightarrow \quad f = 0 \text{ a.e.},$$

not necessarily $f(x) = 0$ at every point.
Baby picture: L^p elements are really equivalence classes. Point values are usually not part of the data.

53. $L^\infty(\Omega)$ and Essential Supremum

$$L^\infty(\Omega) = \{ f : \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty \}.$$

The essential supremum ignores spikes on zero-measure sets:

$$\operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf\{ M \ge 0 : |f(x)| \le M \text{ a.e.} \}.$$

Baby picture: A single infinitely tall spike on dust does not change the L^∞ size.

54. Useful L^p Examples

On $\Omega = (0, 1)$,

$$f(x) = \frac{1}{\sqrt{x}} \quad \Rightarrow \quad f \in L^p(0,1) \iff p < 2.$$

On $\Omega = (1, \infty)$,

$$f(x) = \frac{1}{\sqrt{x}} \quad \Rightarrow \quad f \in L^p(1,\infty) \iff p > 2.$$

Key point: Near 0, the issue is singularity. Near ∞, the issue is tail decay.

Baby picture: The same formula can be integrable near zero but not at infinity, depending on how the power behaves.

55. Hölder, Cauchy–Schwarz, and Minkowski

Hölder:

$$\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

When $p = q = 2$, Hölder becomes Cauchy–Schwarz:

$$\left|\int_\Omega fg\right| \le \|f\|_{L^2}\|g\|_{L^2}.$$

Minkowski:

$$\|f + g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

Baby picture: Hölder controls products. Cauchy–Schwarz is the L^2 version. Minkowski gives the triangle inequality.

56. Fischer–Riesz and the Completion Payoff

For $1 \le p \le \infty$,

$$L^p(\Omega) \text{ is Banach.}$$

In particular,

$$L^2(\Omega) \text{ is Hilbert}$$

with

$$(f, g)_{L^2} = \int_\Omega fg.$$

This is the payoff of Lebesgue integration: it gives complete function spaces where limits of Cauchy sequences stay inside the space.
Baby picture: Lebesgue spaces are closed rooms. Cauchy sequences do not fall through holes.

56a. Dense Does Not Mean Complete

A smooth test-function space such as $C_0^\infty(\Omega)$ can be dense in a larger space but still not complete under that larger norm.
For example, under the L^2 norm, a Cauchy sequence of smooth compactly supported functions may converge in L^2 to a function that is no longer smooth.

$$\overline{C_0^\infty(\Omega)}^{L^2} \subseteq L^2(\Omega),$$

and on many standard domains the closure is all of $L^2(\Omega)$.
Key point: Completing a space means adding the missing limits of its Cauchy sequences.
Baby picture: Smooth functions are like polished stones. If you take limits, the limiting rock may no longer be polished, but it still belongs in the completed room.

57. Pointwise Values Are Usually Not Legal in L^p

A general element of $L^p(\Omega)$ does not have a well-defined value at a single point.
Reason: changing $f(x_0)$ at one point does not change the equivalence class.

$$f \sim g \quad \text{if} \quad f = g \text{ a.e.}$$

Trap: Do not ask for $f(x_0)$ unless you know the L^p class has a special representative, for example a continuous Sobolev representative from an embedding theorem.

57a. Density of Test Functions in L^p

For $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^n$ open, the space of test functions is dense in L^p:

$$\overline{\mathcal{D}(\Omega)}^{L^p(\Omega)} = L^p(\Omega).$$

Consequence: every L^p function can be approximated in L^p-norm by smooth compactly supported functions.
Trap: This fails for $p = \infty$: uniform limits of continuous functions are continuous, so smooth functions cannot approximate arbitrary L^∞ discontinuities.
Baby picture: This density lets us define weak derivatives, traces, and many integral identities by approximation.

57b. Riesz Representation for L^p Dual (Hölder Duality)

For $1 \le p < \infty$ and conjugate exponent $\frac{1}{p} + \frac{1}{q} = 1$ (with $q = \infty$ for $p = 1$), the dual of L^p is isometrically isomorphic to L^q:

$$(L^p(\Omega))' \cong L^q(\Omega), \qquad g \longleftrightarrow F_g, \quad F_g(f) = \int_\Omega fg.$$

Consequences:

- L^p is reflexive for $1 < p < \infty$.
- L^2 is its own dual (Hilbert).
- L^1 has dual L^∞, but L^∞ has a strictly larger dual than L^1 (so L^1 is not reflexive).

Baby picture: Every continuous linear sensor on L^p is integration against a function in the Hölder-conjugate L^q.

Part VIII. Weak Derivatives and Sobolev Spaces

58. Test Functions and Support

$$\mathcal{D}(\Omega) = C_0^\infty(\Omega)$$

is the space of smooth functions with compact support inside Ω.
The support of f is

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

$$\omega \Subset \Omega$$

means ω is compactly contained in Ω.
Baby picture: Test functions are smooth probes that die before touching the boundary.

59. Locally Integrable Functions

$$L^1_{\text{loc}}(\Omega) = \{f : f \in L^1(K) \text{ for every compact } K \Subset \Omega\}.$$

This is the minimal natural class for weak derivatives, because the integrals against compactly supported test functions must make sense.
Baby picture: Locally integrable means integrable on every small safe region inside the domain.

60. Multi-Index Notation

A multi-index is

$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

Then

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Example:

$$D^{(2,3,0)}u = \frac{\partial^5 u}{\partial x_1^2 \partial x_2^3}.$$

Baby picture: Multi-index notation is just a compact bookkeeping device for many partial derivatives.

61. Weak Derivative

A function g is the weak derivative of f with respect to x_i if:

$$\int_\Omega gv = -\int_\Omega f \frac{\partial v}{\partial x_i} \qquad \forall v \in \mathcal{D}(\Omega).$$

More generally, $g = D_w^\alpha f$ if

$$\int_\Omega gv = (-1)^{|\alpha|} \int_\Omega f D^\alpha v \qquad \forall v \in \mathcal{D}(\Omega).$$

Key point: A weak derivative is defined by integration by parts.
Baby picture: Move the derivative from the rough function onto the smooth test function. If the identity always works, the rough function has a weak derivative.

61a. Weak Derivative: Exists vs Fails (HW5 / Integration)

On $(-1, 1)$, let $g_1(x) = 1 - |x|$. Then $g_1 \in W^{1,1}(-1, 1)$ with weak derivative $g_1'(x) = -\text{sign}(x)$ a.e.
By contrast, the step function

$$g_2(x) = \mathbf{1}_{[0,1]}(x)$$

is in $L^1(-1, 1)$ but has *no* weak derivative in $L^1(-1, 1)$: integration by parts against a test function v with $v(0) \neq 0$ would force a δ_0 term, which is not an L^1 function.
Trap: A function can be integrable without having a weak derivative in L^1. Corners compatible with integration by parts (like $1 - |x|$) are allowed; jumps are not.
Baby picture: $1 - |x|$ passes the probe test; a step fails because the probe sees the jump.

62. Sobolev Spaces

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega),\ |\alpha| \leq m\}.$$

For $1 \leq p < \infty$,

$$\|f\|_{W^{m,p}} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p}^p \right)^{1/p}.$$

$$H^m(\Omega) = W^{m,2}(\Omega).$$

Baby picture: Sobolev spaces keep track of both the function and its weak derivatives.

63. H^m Spaces Are Hilbert Spaces

$$H^m(\Omega) = W^{m,2}(\Omega).$$

They carry the inner product

$$(f, g)_{H^m} = \sum_{|\alpha| \leq m} (D^\alpha f, D^\alpha g)_{L^2}.$$

Baby picture: H^m is the Sobolev world where Hilbert geometry works.

64. Density and Approximation

For $1 \le p < \infty$, smooth functions are dense in Sobolev spaces under suitable domain assumptions:

$$C^\infty(\Omega) \cap W^{m,p}(\Omega) \text{ is dense in } W^{m,p}(\Omega).$$

For Lipschitz domains, one often has approximation by functions smooth up to the boundary.
Trap: The case $p = \infty$ is special. Uniform limits of continuous functions stay continuous, so smooth functions cannot approximate arbitrary L^∞ discontinuities.
Baby picture: Density means rough Sobolev functions can be reached as limits of smoother functions. This is why we can prove things first for smooth functions and then pass to limits.

65. Lipschitz Functions and Domains

A Lipschitz function satisfies

$$|f(x) - f(y)| \le M|x - y|.$$

For many reasonable domains,

$$\mathrm{Lip}(\Omega) = W^{1,\infty}(\Omega).$$

A Lipschitz domain has a boundary that can locally be described by Lipschitz graphs.
Baby picture: A Lipschitz boundary may have corners, but not infinitely sharp cusps or fractal garbage. This is exactly the kind of boundary where traces and extensions behave well.

66. Sobolev Embedding Theorem (Three Regimes)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 \le p < \infty$, $m \ge 1$ integer. Sobolev embedding has three regimes governed by the comparison of mp and n:
Subcritical regime ($mp < n$):

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{m}{n}.$$

Critical regime ($mp = n$):

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for every finite } q \in [1, \infty).$$

But the embedding into L^∞ can fail (e.g. $|\ln \|x\||^s$ in $H^1(B_1) \subset \mathbb{R}^2$ for $0 < s < 1/2$).
Supercritical regime ($mp > n$):

$$W^{m,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \qquad \alpha = \min\bigl(m - n/p,\, 1\bigr)^-,$$

giving Hölder-continuous representatives.
Quick checks:

- $H^1 \hookrightarrow C^0$ in 1D ($mp = 2 > 1 = n$).
- $W^{1,3} \hookrightarrow C^0$ in 2D.
- $W^{1,4} \hookrightarrow C^0$ in 3D.

Trap: Embedding arrows point from the stronger (source) space into the weaker/consequence (target) space. Always check which regime mp vs n falls in before claiming a continuous representative.

67. Trace Operator

The trace operator gives boundary values for Sobolev functions:

$$\gamma u = u|_{\partial\Omega}.$$

For smooth functions this is classical. For Sobolev functions, it is defined by density: approximate u by smooth functions u_n, take classical boundary values $u_n|_{\partial\Omega}$, and pass to a limit in a boundary L^p space.
For a Lipschitz domain,

$$\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

is continuous in the trace sense.

Baby picture: Trace is how a rough interior function still has a controlled boundary shadow.

68. Trace Theorem and H_0^1

Trace inequality (Lipschitz domain). Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $1 \le p < \infty$. There is a constant $C = C(\Omega, p)$ such that for every $v \in C^1(\overline{\Omega})$,

$$\|v\|_{L^p(\partial\Omega)} \le C\, \|v\|_{L^p(\Omega)}^{1-1/p}\, \|v\|_{W^{1,p}(\Omega)}^{1/p}.$$

The trace map extends by density to a continuous linear operator

$$\gamma : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega).$$

Zero-trace Sobolev space.

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \gamma u = 0\} = \overline{\mathcal{D}(\Omega)}^{W^{1,p}},$$

and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.
Normal trace (corollary). A normal-trace operator γ_1 can be defined on $H^2(\Omega)$ with values in $L^2(\partial\Omega)$.
Baby picture: H_0^1 means zero boundary value in the trace sense, not necessarily point-by-point classical zero.

68a. Green's Formula in H^1 (HW5)

For a bounded Lipschitz domain Ω and $u, v \in H^1(\Omega)$,

$$\int_\Omega v\, \partial_i u = -\int_\Omega (\partial_i v)\, u + \int_{\partial\Omega} u\, v\, n_i\, dS.$$

Proof pattern (density). Define the three bilinear pieces

$$B_1(u,v) = \int_\Omega v\, \partial_i u, \quad B_2(u,v) = -\int_\Omega (\partial_i v)\, u, \quad B_3(u,v) = \int_{\partial\Omega} u\, v\, n_i.$$

Show each B_j is continuous on $H^1 \times H^1$ (Hölder + trace for B_3). The identity holds for $u, v \in C^1(\overline{\Omega})$ by classical integration by parts; extend to all of H^1 by density.
Key point: Green's formula is not a pointwise rule; it is a limit of smooth identities plus continuity of bilinear forms.
Baby picture: Prove continuity of each term, then density does the rest.

Part IX. HW5 Addendum: C^1, Weak Derivatives, and Energy Norms

HW5-1. Classical Smoothness: C^0, C^1, and C^k

$$C^0(\Omega) = \{u : \Omega \to \mathbb{R} : \ u \text{ is continuous}\}.$$

$$C^1(\Omega) = \{u : \Omega \to \mathbb{R} : \ u \text{ is continuous and } \nabla u \text{ is continuous}\}.$$

More generally,

$$C^k(\Omega) = \{u : \text{all derivatives up to order } k \text{ are continuous}\}.$$

Baby picture: C^0 means no jumps in the function. C^1 means no jumps in the slope. C^k means several layers of smoothness.

HW5-2. Why C^1 Is Often Too Strong

For many PDEs and FEM approximations, requiring $u \in C^1$ is too restrictive.
Reasons:

- true solutions may have corners or weak singularities,
- material coefficients may be discontinuous,
- FEM functions are often only piecewise smooth,
- C^1-conforming elements are harder to build than C^0-conforming elements.

Baby picture: C^1 asks the solution to wear a tuxedo. PDE solutions often show up in work clothes.

HW5-3. $C^1(\overline{\Omega})$ Can Be an Inner-Product Space but Still Not Hilbert

For the elliptic energy product

$$\langle u, v\rangle = \int_\Omega k\nabla u \cdot \nabla v\, dx + \int_\Omega suv\, dx,$$

$V = C^1(\overline{\Omega})$ may be an inner-product space.
But it is not necessarily complete under the induced norm. Therefore Riesz does *not* automatically guarantee a solution $u \in C^1(\overline{\Omega})$ to

$$\langle u, v\rangle = F(v) \qquad \forall v \in V.$$

Key point: Inner product alone is not enough. Hilbert means inner product plus completeness.
Concrete incompleteness witness (Perspective). On $(0,1)$, with energy inner product from $\int (u')^2$, the sequence

$$u_\varepsilon(x) = x \arctan\!\left(\frac{x}{\varepsilon}\right) - \frac{\varepsilon}{2}\log(x^2+\varepsilon^2)$$

is Cauchy in the energy norm, but its limit is not in $C^1([0,1])$.
Baby picture: C^1 has a ruler and angles, but the room may still have holes. Sobolev spaces patch the holes.

HW5-4. Discontinuous Coefficients Break Classical C^1 Thinking

For

$$-\nabla \cdot (k\nabla u) + su = f,$$

if k jumps across an interface, then u may fail to be C^1.
In one dimension, if

$$k(x) = \begin{cases} 2, & x \le 1/2, \\ 1, & x > 1/2, \end{cases}$$

the physically meaningful condition is flux continuity:

$$(ku')^+ = (ku')^-.$$

So u' itself may jump even when the flux is continuous.
Baby picture: The material changes suddenly, so the slope may kink. The flux behaves correctly, but C^1 smoothness is too strict.

HW5-5. Weak Derivative: The HW5 Core Idea

For $u \in L^1_{\text{loc}}(\Omega)$, we say g is the weak derivative of u in direction x_i if

$$\int_\Omega gv\, dx = -\int_\Omega u\frac{\partial v}{\partial x_i}\, dx \qquad \forall v \in C_0^\infty(\Omega).$$

This comes from integration by parts, but the derivative is moved from u onto the test function v.
Baby picture: Weak derivative means: instead of checking the slope point-by-point, we check whether the function behaves correctly against every smooth probe.

HW5-6. Classical Derivative Implies Weak Derivative

If $u \in C^1(\Omega)$, then its classical derivative is also its weak derivative.

$$D_i^{\text{weak}} u = D_i^{\text{classical}} u.$$

So weak derivatives extend classical derivatives; they do not contradict them.
Baby picture: Weak derivative is backward-compatible: if the usual slope exists, the weak slope agrees with it.

HW5-7. $H^1(\Omega)$ as Finite-Energy Space

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}.$$

The norm is

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2\right)^{1/2}.$$

Baby picture: H^1 means the function has finite size and finite slope energy.

HW5-8. $H_0^1(\Omega)$: Boundary-Tied Energy Space

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{H^1}.$$

Equivalently, for nice domains:

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \gamma_0 u = 0 \text{ on } \partial\Omega\}.$$

Baby picture: H_0^1 means the function is tied down to zero on the boundary in the trace sense, not necessarily in a classical pointwise sense.

HW5-9. Poincaré Inequality

For $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}.$$

Key point: This says that on H_0^1, the slope controls the whole function.
Baby picture: If the boundary is pinned to zero, the function cannot float upward as a constant. To become large inside, it must build slope.

HW5-10. Equivalent Norms on $H_0^1(\Omega)$

The usual H^1-norm is

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{1/2}.$$

On $H_0^1(\Omega)$, Poincaré implies:

$$\|u\|_{H^1(\Omega)} \sim \|\nabla u\|_{L^2(\Omega)}.$$

That is, there exist constants $c, C > 0$ such that

$$c\|u\|_{H^1} \le \|\nabla u\|_{L^2} \le C\|u\|_{H^1}.$$

Baby picture: On H_0^1, the gradient alone is a complete ruler because the boundary condition kills the constant-function ambiguity.

HW5-11. Energy Inner Product and Coercivity

For elliptic PDEs, we often use

$$a(u, v) = \int_\Omega k\nabla u \cdot \nabla v \, dx + \int_\Omega s u v \, dx.$$

If $k(x) \ge k_{\min} > 0$ and $s(x) \ge 0$, then

$$a(u, u)$$

controls the energy of u. With the right boundary condition, it controls the H^1-norm.
Baby picture: Coercivity means the energy cannot be small unless the function itself is small. It prevents the PDE from having floppy zero-energy modes.

HW5-12. Common HW5 Trap: Forgetting the Boundary Condition

On $H^1(\Omega)$, the gradient seminorm

$$\|\nabla u\|_{L^2}$$

does not control constants.
Example:

$$u(x) = 1 \quad \Rightarrow \quad \nabla u = 0$$

but

$$u \neq 0.$$

On $H_0^1(\Omega)$, constants are killed by the zero-boundary condition, so Poincaré saves us.
Trap: Do not claim $\|\nabla u\|_{L^2}$ is a norm on all of H^1. It becomes equivalent to the H^1-norm only with boundary restrictions such as H_0^1.

Part X. PDE Weak Formulation and FEM

72. Strong PDE Form

A model elliptic PDE:

$$-\nabla \cdot (k\nabla u) + su = f \qquad \text{in } \Omega,$$

with boundary condition:

$$u = 0 \qquad \text{on } \partial\Omega.$$

Classically, this requires u to be smooth enough.
Baby picture: Strong form asks the equation to hold point by point. This is often too strict.

73. Weak / Variational Form

Multiply by a test function v, integrate, and integrate by parts.
Find $u \in H_0^1(\Omega)$ such that:

$$a(u, v) = F(v) \qquad \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_\Omega k\nabla u \cdot \nabla v \, dx + \int_\Omega suv \, dx,$$

and

$$F(v) = \int_\Omega fv \, dx.$$

Baby picture: Weak form says: the PDE is true when tested against every allowable probe.

74. Energy Norm

If $a(\cdot, \cdot)$ is an inner product, it induces the energy norm:

$$\|u\|_a = \sqrt{a(u, u)}.$$

Baby picture: Energy norm measures error in the physically meaningful energy of the PDE.

74a. Bounded Bilinear Form ⇔ Continuous (HW5)

Let V be a normed space and $a : V \times V \to \mathbb{R}$ bilinear. Then

$$\exists M > 0 : \ |a(u, v)| \le M\|u\|\|v\| \quad \iff \quad a \text{ is continuous on } V \times V.$$

Use. Every variational estimate in HW5–HW7 starts by proving $|a(u, v)| \le M\|u\|\|v\|$, then invoking density or Lax–Milgram.
Baby picture: The inequality $\|a(u, v)\| \le M\|u\|\|v\|$ is the whole continuity proof.

75. Lax–Milgram Lemma

Let V be a Hilbert space and $a : V \times V \to \mathbb{R}$ a bilinear form satisfying:
- **continuity:** $|a(u, v)| \le M\,\|u\|_V\|v\|_V$ for all $u, v \in V$,
- **coercivity:** $a(u, u) \ge \alpha\,\|u\|_V^2$ for all $u \in V$, with $\alpha > 0$.

Then for every $F \in V'$ there is a unique $u \in V$ with

$$a(u, v) = F(v) \quad \forall v \in V,$$

and

$$\|u\|_V \le \frac{1}{\alpha}\,\|F\|_{V'}.$$

Symmetric case. If a is symmetric, $a(\cdot, \cdot)$ is an inner product, and Lax–Milgram reduces to the Riesz representation theorem in the energy inner product.
Baby picture: Coercivity is the energy floor that turns a continuous bilinear form into a usable inner product for existence.

75a. Finite-Dimensional Subspaces Are Complete

If S_h is a finite-dimensional subspace of a normed space, then S_h is complete.
Therefore, once an inner product is placed on S_h,

$$S_h \text{ is Hilbert.}$$

This is why the discrete FEM problem can have a clean existence argument even when the larger classical space is not complete.

$$S_h \text{ finite-dimensional} \Rightarrow S_h \text{ complete} \Rightarrow S_h \text{ Hilbert} \Rightarrow \exists! u_h.$$

Baby picture: Finite-dimensional rooms have no hidden infinite-dimensional holes.

75b. Completion Logic for the Continuous Problem

If the original classical space $V = C^1(\overline{\Omega})$ is incomplete, the equation

$$\langle u, v\rangle = F(v) \qquad \forall v \in V$$

may not have a solution in V.
The fix is to complete the space in the right norm. The completion is no longer a classical C^1 space; it becomes a Sobolev-type space such as H^1 or H^1_0.

$$\text{complete the energy space} \Rightarrow \text{Hilbert space} \Rightarrow \text{Riesz / Lax–Milgram applies.}$$

Baby picture: The solution may live exactly at a missing limit point of the classical space. Completing the space adds that point back in.

75c. Banach–Nečas–Babuška (BNB) Theorem

Let V, W be reflexive Banach spaces and $a : V \times W \to \mathbb{R}$ a continuous bilinear form. The variational problem

$$\text{find } u \in V : \quad a(u, w) = F(w) \quad \forall w \in W$$

is well-posed for every $F \in W'$ if and only if both:

- **inf-sup condition:** there exists $\beta > 0$ with

$$\inf_{v \in V \setminus \{0\}} \sup_{w \in W \setminus \{0\}} \frac{a(v, w)}{\|v\|_V \|w\|_W} \geq \beta,$$

- **non-degeneracy:** for every nonzero $w \in W$, there exists $v \in V$ with $a(v, w) \neq 0$.

The solution satisfies $\|u\|_V \leq \beta^{-1} \|F\|_{W'}$.
Coercive $\Rightarrow$ BNB. Lax–Milgram is the special case where $V = W$ and the diagonal coercivity $a(v, v) \geq \alpha \|v\|^2$ implies $\beta \geq \alpha$.
Discrete BNB. On a finite-dimensional subspace $V_h \subset V$ paired with $W_h \subset W$, the discrete problem is well-posed iff

$$\beta_h := \inf_{v_h \in V_h \setminus \{0\}} \sup_{w_h \in W_h \setminus \{0\}} \frac{a(v_h, w_h)}{\|v_h\| \|w_h\|} \geq \beta_0 > 0$$

uniformly in h.
Baby picture: BNB is Lax–Milgram without symmetry between trial and test spaces, and without diagonal coercivity. Inf-sup is the right replacement.

76. FEM Subspace

Choose a finite-dimensional subspace:

$$S_h \subset H_0^1(\Omega).$$

Find $u_h \in S_h$ such that:

$$a(u_h, v_h) = F(v_h) \qquad \forall v_h \in S_h.$$

Since S_h is finite-dimensional, it is complete.
Baby picture: FEM replaces the infinite-dimensional solution space with a finite-dimensional computable one.

77. FEM as Projection

If exact u satisfies:

$$a(u, v) = F(v) \qquad \forall v \in H_0^1,$$

and $u_h \in S_h$ satisfies:

$$a(u_h, v_h) = F(v_h) \qquad \forall v_h \in S_h,$$

then:

$$a(u - u_h, v_h) = 0 \qquad \forall v_h \in S_h.$$

Thus u_h is the orthogonal projection of u onto S_h in the energy inner product.
Baby picture: The FEM solution is the best shadow of the true solution on the FEM space.

77a. Céa's Lemma (Quasi-Optimality)

Under Lax–Milgram hypotheses (continuity M, coercivity α), the Galerkin solution $u_h \in S_h$ satisfies

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in S_h} \|u - v_h\|_V.$$

In the symmetric case the constant improves to $\sqrt{M/\alpha}$ in the energy norm.
Interpretation: the discrete error is at most M/α times the *best approximation* error.
Key point: Céa reduces convergence rate analysis to interpolation theory: control $\inf_{v_h} \|u - v_h\|_V$.
Baby picture: Galerkin is quasi-optimal: only off the best approximation by a constant.

78. Why Classical C^1 Spaces Are Not Enough

Classical spaces such as $C^1(\Omega)$ are often:

- too small,
- not complete under the energy norm,
- incompatible with rough coefficients,
- too restrictive for weak PDE solutions.

Sobolev spaces fix this by allowing rougher functions while keeping enough derivative structure.
Baby picture: Classical functions are too fancy. PDE solutions often wear work boots, not tuxedos. Sobolev spaces let them in.

77b. Compact Embedding $X \subset\subset Y$ (HW6)

We write $X \subset\subset Y$ if

- $X \hookrightarrow Y$ continuously (there exists C with $\|u\|_Y \le C\|u\|_X$),
- the inclusion maps every bounded set in X to a relatively compact set in Y.

Example. For bounded Lipschitz Ω, Rellich–Kondrachov gives $H^1(\Omega) \subset\subset L^2(\Omega)$.
Key point: Compact embedding turns weak limits into strong limits in the target space — the engine behind Robin coercivity and Fredholm arguments.
Baby picture: Continuous inclusion plus bounded sets becoming precompact in the target.

78a. Rellich–Kondrachov Compact Embedding

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$,

$$H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega),$$

i.e. the inclusion is **compact**. More generally, for Ω bounded Lipschitz and $1 \le p < \infty$ with $mp < n$, $W^{m,p}(\Omega)$ embeds compactly into $L^q(\Omega)$ for any q with $1/q > 1/p - m/n$.
Key point: Bounded sets in H^1 have L^2-convergent subsequences. This is the compactness engine that drives Fredholm alternative and existence in non-coercive problems.
Baby picture: Going up in Sobolev regularity and dropping back to L^p is rewarded with compactness.

78b. Aubin–Nitsche Duality and L^2 Estimates

For a symmetric coercive problem with $u \in H_0^1(\Omega)$ and FEM solution $u_h \in S_h \subset H_0^1$ on a quasi-uniform mesh of size h, assuming H^2-regularity of the dual problem,

$$\|u - u_h\|_{L^2(\Omega)} \le C\, h\, \|u - u_h\|_{H^1(\Omega)}.$$

Combined with Céa and standard interpolation, this gives

$$\|u - u_h\|_{L^2} = O(h^{k+1}), \qquad \|u - u_h\|_{H^1} = O(h^k),$$

when using degree-k piecewise polynomials.
Baby picture: Solving an auxiliary dual problem buys one extra power of h in the L^2 error.

78c. Gårding's Inequality

A continuous bilinear form $a : H \times H \to \mathbb{R}$ on a Hilbert space $H \hookrightarrow L^2(\Omega)$ satisfies Gårding's inequality if there exist $\alpha > 0$ and $\lambda \ge 0$ such that

$$a(u,u) + \lambda\, \|u\|_{L^2}^2 \ge \alpha\, \|u\|_H^2 \qquad \forall u \in H.$$

This is "coercivity up to a compact lower-order shift." Combined with a compact embedding $H \hookrightarrow\hookrightarrow L^2$, it lets non-coercive problems (e.g. indefinite Helmholtz, convection-dominated) still fall under Fredholm-type theory.
Baby picture: Gårding-coercive forms are coercive after subtracting a compactly bounded perturbation.

78d. Fredholm Alternative

Let H be a Hilbert space, $T : H \to H$ a compact linear operator, and I the identity. Then exactly one of the following holds:

1. For every $f \in H$, the equation $u - Tu = f$ has a unique solution $u \in H$.
2. The homogeneous equation $u - Tu = 0$ has a nontrivial solution, and $u - Tu = f$ is solvable if and only if $f \perp \ker(I - T^*)$.

Applied to PDEs. A Gårding-coercive variational problem can be cast as $u = Tu + g$ for some compact T, so existence and uniqueness become a finite-dimensional kernel question.
Baby picture: For compact perturbations of identity, "existence for all f" and "some kernel exists" are mutually exclusive — you choose by checking the homogeneous problem.

78e. Essential vs Natural Boundary Conditions

Essential BC. A condition imposed by membership in the trial or test space, e.g. Dirichlet $u|_{\partial\Omega} = g$ via the affine space $H^1_g(\Omega)$.
Natural BC. A condition that arises automatically from integration by parts in the weak form, e.g. Neumann $k\partial_n u = h$ on $\partial\Omega$ in the bilinear form

$$a(u, v) = \int_\Omega k\nabla u \cdot \nabla v + \int_\Omega suv, \qquad F(v) = \int_\Omega fv + \int_{\partial\Omega} hv.$$

Trap: Neumann conditions are never imposed by membership in H^1; they are encoded in the load functional and the bilinear form.
Baby picture: Essential is who you let into the space. Natural falls out for free when you integrate by parts.

78e2. Non-Homogeneous Dirichlet by Lifting (HW6)

For prescribed boundary data $\bar{u} \in H^{1/2}(\partial\Omega)$ (or $\bar{u}$ smooth), seek $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = \bar{u}$ solving

$$a(u, v) = F(v) \qquad \forall v \in H^1_0(\Omega).$$

Lifting. Write $u = \bar{u} + w$ with $w \in H^1_0(\Omega)$. The problem for w has homogeneous Dirichlet data.
Hadamard stability. Small changes in f and in $\bar{u}$ produce small changes in u:

$$\|u\|_{H^1} \le C\big(\|F\|_{(H^1_0)'} + \|\bar{u}\|_{H^{1/2}(\partial\Omega)}\big).$$

Baby picture: Fix the boundary first, then solve for the correction in H^1_0.

78f. Pure Neumann Problem: Compatibility Condition

For the pure Neumann problem

$$-\nabla \cdot (k\nabla u) = f \text{ in } \Omega, \qquad k\partial_n u = h \text{ on } \partial\Omega,$$

the variational form on $V = H^1(\Omega)$ has *constants in its kernel.*
Solvability (Fredholm). A solution exists iff the data satisfy the compatibility condition

$$\int_\Omega f\,dx + \int_{\partial\Omega} h\,dS = 0.$$

The solution is unique up to an additive constant; this is fixed by, e.g., $\int_\Omega u = 0$.
Baby picture: Without Dirichlet pinning, you need to enforce a global balance and then pin the average.

78g. Robin BC and Coercivity by Compactness (HW6)

Consider

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + k \int_{\partial\Omega} u \, v \, dS, \qquad k > 0,$$

on $H^1(\Omega)$. The associated strong form is $-\Delta u + ku = 0$ with Robin condition $\partial_n u + ku = 0$ on $\partial\Omega$.

Coercivity proof (contradiction + compactness).

1. If $a(u,u) = 0$, normalize u so $\|u\|_{H^1} = 1$.
2. Extract $u_n \rightharpoonup u$ weakly in H^1 (bounded sequence).
3. Rellich–Kondrachov: $u_n \to u$ strongly in $L^2(\Omega)$.
4. Pass to the limit in $a(u_n, u_n)$: interior term gives $\|\nabla u\|_{L^2} = 0$; boundary term gives $k\|u\|_{L^2(\partial\Omega)} = 0$, hence $u = 0$.

Key point: Weak convergence in H^1 plus compact $H^1 \hookrightarrow L^2$ upgrade boundary terms to strong limits.

Baby picture: Robin term on the boundary is coercive because a weakly convergent unit sphere cannot hide mass on $\partial\Omega$.

Part XI. Optional Bigger-Picture Topology Addendum

86. Topological Space

A topology on a set X is a collection $\mathcal{T}$ of subsets called open sets such that:

- $\emptyset, X \in \mathcal{T}$,
- arbitrary unions of open sets are open,
- finite intersections of open sets are open.

Baby picture: A topology is a rulebook for what counts as open, hence what counts as near.

87. Topological Vector Space

A topological vector space is both a vector space and a topological space, with addition and scalar multiplication continuous:

$$(x,y) \mapsto x + y, \qquad (\alpha, x) \mapsto \alpha x.$$

Every normed space is a topological vector space under its norm topology.

Baby picture: A topological vector space is a linear world where the operations behave continuously with respect to the chosen notion of closeness.

88. Nets: Why Sequences Are Not Always Enough

In metric spaces, closure can be detected by sequences. In general topological spaces, sequences may miss limit points.

A net generalizes a sequence by replacing the index set $\mathbb{N}$ with a directed set.

$$x_\alpha \to x \quad \Longleftrightarrow \quad \text{eventually } x_\alpha \text{ lies in every neighborhood of } x.$$

Baby picture: A sequence is a one-lane approach to a point. A net is a multi-lane approach that can enter every possible neighborhood system.

89. Baire Category Theorem: Completeness Has Consequences

A complete metric space cannot be written as a countable union of nowhere dense sets.

Equivalently, in a complete metric space, the complement of a countable union of nowhere dense sets is dense.

Baby picture: A complete space is too large and solid to be built from countably many thin dusty layers.

90. Do Not Overload the Main Course Story

Topological vector spaces, nets, and Baire category are part of the larger functional-analysis universe.
For the current PDE and FEM story, the main line is still:

$$\text{normed space} \to \text{complete space} \to \text{Hilbert geometry} \to \text{weak PDE/FEM}.$$

Trap: Use the topology addendum as context, not as the main computation engine unless the homework explicitly asks for it.

Part XII. FEM Spaces and Interpolation

91. Ciarlet Finite Element

A **finite element** is a triple

$$(K,\, P_K,\, \Sigma_K)$$

where

- $K \subset \mathbb{R}^n$ is a closed connected set with nonempty interior and Lipschitz boundary (e.g. simplex, hexahedron),
- P_K is a finite-dimensional space of functions on K (typically polynomials),
- $\Sigma_K = \{\sigma_1, \dots, \sigma_d\} \subset P_K'$ is a basis of degrees of freedom (DoFs), $d = \dim P_K$, such that P_K is **unisolvent**: every $p \in P_K$ is uniquely determined by $(\sigma_i(p))_{i=1}^d$.

The associated **nodal basis** $\{\theta_i\} \subset P_K$ satisfies $\sigma_i(\theta_j) = \delta_{ij}$.
Baby picture: Element = geometry + local space + linear sensors. Unisolvence makes the linear system invertible.

91a. Unisolvence Counterexample: Q_1 with Edge-Midpoint DoFs (HW7)

On $K = [-1,1]^2$, let $P_K = \mathbb{Q}_1(K)$ (bilinear functions) but take DoFs as values at the four edge midpoints (not vertices).
Then two different bilinear functions can share the same four midpoint values, so the map $p \mapsto (\sigma_i(p))$ is *not* injective.
Trap: A sensible-looking DoF set does not automatically define a finite element. Unisolvence must be checked.
Baby picture: Edge midpoints do not determine a unique bilinear polynomial on the square.

92. Affine Equivalence and Master Element

Two finite elements (K, P_K, Σ_K) and $(\hat{K}, \hat{P}, \hat{\Sigma})$ are **affine equivalent** if there exists an affine bijection $F_K : \hat{K} \to K$, $F_K(\hat{x}) = B_K\hat{x} + b_K$, such that $P_K = \{\hat{p} \circ F_K^{-1} : \hat{p} \in \hat{P}\}$ and the DoFs transform accordingly.
Master element scheme. Implement everything on a fixed reference ("master") element $\hat{K}$; map quadrature, basis, and integrals to physical K via F_K.
Mesh geometry constants. For the affine map,

$$h_K = \operatorname{diam}(K), \qquad \rho_K = \sup\{\operatorname{diam}(B) : B \subset K \text{ ball}\}.$$

Baby picture: Affine equivalence lets you analyze one master element and transport all estimates to every physical cell.

93. Local Interpolation Operator and Estimate

The **local interpolation operator** $I_K : C^0(K) \to P_K$ is

$$I_K v = \sum_{i=1}^{d} \sigma_i(v)\, \theta_i, \qquad \sigma_i(I_K v) = \sigma_i(v).$$

Local error estimate. For an affine family of elements with $P_K \supset \mathbb{P}_k(K)$, and $v \in W^{k+1,p}(K)$ with the DoFs σ_i

continuous on $W^{k+1,p}(K) \hookrightarrow C^0$,

$$|v - I_K v|_{W^{m,p}(K)} \le C\,\frac{h_K^{k+1}}{\rho_K^m}\,|v|_{W^{k+1,p}(K)}.$$

Baby picture: Local interpolation error scales as h^{k+1} in the L^p norm, h^{k+1-m} in the $W^{m,p}$ semi-norm, on a shape-regular element.

94. Shape-Regular and Quasi-Uniform Meshes

A family of meshes $\{\mathcal{T}_h\}$ is:

- **shape regular** if there is $\kappa \ge 1$ such that for all K and all h

$$\frac{h_K}{\rho_K} \le \kappa.$$

- **quasi-uniform** if it is shape regular and there is $c > 0$ such that

$$\min_K h_K \ge c \max_K h_K = c\,h.$$

Trap: Without shape regularity the local error estimate constants blow up; without quasi-uniformity, inverse inequalities are not available.
Baby picture: Shape regularity = no degenerate slivers. Quasi-uniformity = no extreme size variation.

95. Global Interpolation and Estimate

The global interpolant $I_h : C^0(\overline{\Omega}) \to V_h$ is defined elementwise: $(I_h v)|_K = I_K(v|_K)$.
Global estimate. On a shape-regular affine family with $\mathbb{P}_k \subset P_K$,

$$\|v - I_h v\|_{W^{m,p}(\Omega)} \le C\,h^{k+1-m}\,|v|_{W^{k+1,p}(\Omega)}, \qquad 0 \le m \le k+1.$$

Combined with Céa's lemma, this yields the standard FEM convergence rates.
Baby picture: Glue local estimates over the mesh, then take an ℓ^p sum to get the global rate.

95a. Density of P_1 FEM Spaces in L^2 and H^1 (HW7)

On $(0,1)$, let V_n be continuous piecewise linear functions on the uniform mesh with spacing $h_n = 2^{-n}$. Then

$$\overline{\bigcup_{n\ge 1} V_n}^{L^2(0,1)} = L^2(0,1), \qquad \overline{\bigcup_{n\ge 1} V_n}^{H^1(0,1)} = H^1(0,1).$$

Proof idea.

1. Approximate $v \in H^1$ by smooth v_ε (density of C^∞).
2. For fixed n, $I_n v_\varepsilon \in V_n$ agrees with v_ε except on finitely many cells; on each bad cell use the local P_1 interpolation estimate.
3. Let $\varepsilon \to 0$, then choose n large so the FEM error is small.

Key point: The union of all P_1 spaces is dense — Galerkin approximation is built on a hierarchy that actually fills H^1.
Baby picture: Smooth first, then one-element correction on the mesh where smooth and piecewise linear differ.

96. Inverse Inequalities

On a **quasi-uniform** affine family with $P_K \subset \mathbb{P}_k$,

$$\|v_h\|_{W^{m,p}(K)} \le C\,h_K^{\ell-m+n(1/p-1/q)}\,\|v_h\|_{W^{\ell,q}(K)}, \qquad v_h \in P_K,$$

for $0 \le \ell \le m$ and $1 \le q, p \le \infty$.
Most common cases.

$$\|\nabla v_h\|_{L^2(\Omega)} \le C\,h^{-1}\,\|v_h\|_{L^2(\Omega)}, \qquad \|v_h\|_{L^\infty(\Omega)} \le C\,h^{-n/2}\,\|v_h\|_{L^2(\Omega)}.$$

Baby picture: Inside the finite-dimensional FEM space, you may move "downward" in derivative order at the cost of an h^{-1} blow-up factor per derivative.

97. Conformity of FEM Spaces

A finite element space V_h is **conforming** in V if $V_h \subset V$.
Typical conformity rules.

- Piecewise polynomials with continuous global value $\Rightarrow V_h \subset W^{1,p}(\Omega)$.
- Piecewise polynomials with continuous global value and gradient (e.g. Argyris) $\Rightarrow V_h \subset W^{2,p}(\Omega)$.
- Piecewise polynomials without inter-element continuity (DG) $\Rightarrow V_h \not\subset W^{1,p}$; need broken norms.

Baby picture: Need H^1 globally? Then need continuity across faces. Need H^2? Then continuity of value *and* gradient.

97a. Hermite Element May Fail Global C^1 Conformity (HW7)

A triangular Hermite element can enforce value and normal derivative on each edge *locally*, but on a conforming mesh the global interpolant need not be C^1 across element interfaces.
Trap: Local C^1 data do not guarantee a globally C^1 conforming space. Argyris-type elements are needed for H^2 conformity.
Baby picture: Matching slopes on every shared edge is a global compatibility condition, not automatic from local Hermite data.

97b. Nodal-Spike Sequence: L^p vs H^1 Convergence (HW7)

Build $f_n \in V_n$ (P_1 on $(0,1)$) with nodal values alternating $\pm 2^{-n}$ at mesh points. Then:

- $f_n \to 0$ in $L^p(0,1)$ for every $p < \infty$ (spikes shrink),
- $\|f_n'\|_{L^2}$ does not stay bounded, so (f_n) is not Cauchy in H^1,
- hence no strong or weak H^1 limit exists.

Trap: FEM functions can look tame in L^2 while oscillating wildly in derivative norm. Convergence in V is a separate question from pointwise or L^p convergence.
Baby picture: Shrinking spikes in value can still blow up slope energy.

98. Deny–Lions Lemma

For a polynomial-reproducing operator $\Pi : W^{k+1,p}(\hat{K}) \to \mathbb{P}_k(\hat{K})$ on the master element with $\Pi p = p$ for $p \in \mathbb{P}_k$,

$$\|v - \Pi v\|_{W^{k+1,p}(\hat{K})} \le C\,|v|_{W^{k+1,p}(\hat{K})}.$$

Key point: This is the abstract polynomial-quotient inequality that converts "polynomials are reproduced" into the local interpolation estimate via the master element.
Baby picture: Deny–Lions is why interpolation error only sees the highest derivatives of v — the rest is killed by polynomial reproduction.

Part XIII. Elasticity, Mixed Methods, and Stokes

99. Linear Elasticity: Strong and Weak Form

Kinematics. Displacement $u : \Omega \to \mathbb{R}^n$, strain

$$\varepsilon(u) = \tfrac{1}{2}\big(\nabla u + \nabla u^\top\big).$$

Constitutive law (isotropic). Lamé parameters $\lambda, \mu > 0$,

$$\sigma(u) = 2\mu\,\varepsilon(u) + \lambda\,\mathrm{tr}\big(\varepsilon(u)\big)I.$$

Strong form (equilibrium).

$$-\nabla\cdot\sigma(u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_D, \qquad \sigma(u)\cdot n = g \text{ on } \Gamma_N.$$

Weak form. Find $u \in V = \{v \in H^1(\Omega)^n : v|_{\Gamma_D} = 0\}$ with

$$a(u,v) = \int_\Omega \sigma(u) : \varepsilon(v) = \int_\Omega 2\mu\,\varepsilon(u) : \varepsilon(v) + \lambda\,(\nabla\cdot u)(\nabla\cdot v) = F(v).$$

Baby picture: Linear elasticity = scalar Poisson story upgraded to vector fields, with symmetric gradient ε replacing ∇u.

100. Korn's Inequality

Korn's first inequality. For $v \in H_0^1(\Omega)^n$ (or $\Gamma_D \neq \emptyset$),

$$\|\varepsilon(v)\|^2_{L^2(\Omega)} \geq C_K\,\|\nabla v\|^2_{L^2(\Omega)}.$$

Korn's second inequality (general H^1). For $v \in H^1(\Omega)^n$,

$$\|\nabla v\|^2_{L^2} \leq C_K\big(\|\varepsilon(v)\|^2_{L^2} + \|v\|^2_{L^2}\big).$$

Key point: Korn turns control of the symmetric gradient into control of the full gradient. It is the coercivity engine for elasticity.
Trap: Korn requires a nontrivial Dirichlet portion (or a quotient by rigid-body motions). Otherwise the kernel contains all infinitesimal rigid motions.
Baby picture: The symmetric part of the gradient alone controls the full gradient up to lower-order terms.

101. Well-Posedness of Linear Elasticity

Under Korn's first inequality and $\Gamma_D \neq \emptyset$, the bilinear form for linear elasticity is continuous and coercive:

$$a(v,v) \geq 2\mu\,C_K\,\|v\|^2_{H^1}.$$

By Lax–Milgram, the elastic weak problem has a unique solution $u \in V$, with

$$\|u\|_V \leq \frac{1}{2\mu\,C_K}\,\|F\|_{V'}.$$

Baby picture: Coercivity is granted by Korn plus a small piece of Dirichlet boundary.

102. Volumetric Locking

In nearly incompressible elasticity ($\lambda \gg \mu$, i.e. Poisson ratio $\to 1/2$), low-order conforming displacement FEM exhibits **volumetric locking**:

- the discrete kernel of $\nabla\cdot$ becomes too small,
- the FEM solution becomes artificially stiff,

- the constant in Lax–Milgram blows up as $\lambda \to \infty$ unless the inf-sup-stable mixed formulation is used.

Trap: Pure displacement P1 elements lock in the incompressible limit. Switch to mixed (u, p) formulations or higher-order conforming spaces with bubbles.

Baby picture: Lambda-driven stiffness reveals a hidden saddle-point: the divergence constraint must be solved, not penalized.

103. Mixed (u,p) Formulation and Saddle Point

Write incompressible elasticity / Stokes as: find $(u, p) \in V \times Q$ with

$$a(u, v) + b(v, p) = F(v) \quad \forall v \in V,$$

$$b(u, q) = G(q) \quad \forall q \in Q,$$

where typically

$$a(u, v) = \int_\Omega 2\mu\, \varepsilon(u) : \varepsilon(v), \qquad b(v, q) = -\int_\Omega q\, \nabla \cdot v.$$

The pair (u, p) is a critical point (saddle point) of the Lagrangian

$$\mathcal{L}(v, q) = \tfrac{1}{2} a(v, v) - F(v) + b(v, q) - G(q).$$

Baby picture: Mixed methods upgrade "minimization with hard constraint" into a coupled (u, p) system.

104. Inf-Sup / Babuška–Brezzi Condition

The mixed system is well-posed iff:

- **coercivity on the kernel:** there is $\alpha > 0$ with $a(v, v) \geq \alpha \|v\|_V^2$ for all $v \in \ker B$,
- **inf-sup (LBB) condition:** there is $\beta > 0$ with

$$\inf_{q \in Q \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta.$$

Discrete LBB. The discrete pair (V_h, Q_h) must satisfy a uniform inf-sup:

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \beta_0 > 0$$

independent of h.

Trap: Equal-order (P_1, P_1) pairs violate discrete LBB and produce checkerboard pressure modes. Use Taylor–Hood $(P_2–P_1)$, mini element (P_1 bubble–P_1), Crouzeix–Raviart, or stabilized formulations.

Baby picture: Inf-sup says: the pressure can really push the velocity around. If it cannot, you have spurious pressure modes.

105. Abstract Saddle-Point Well-Posedness

Let V, Q be Hilbert, $a : V \times V \to \mathbb{R}$ continuous bilinear, $b : V \times Q \to \mathbb{R}$ continuous bilinear, and define

$$\ker B = \{v \in V : b(v, q) = 0\ \forall q \in Q\}.$$

If a is coercive on $\ker B$ with constant α, and b satisfies the inf-sup condition with constant β, then for every $(F, G) \in V' \times Q'$ there is a unique $(u, p) \in V \times Q$ solving the mixed system, and

$$\|u\|_V + \|p\|_Q \leq C(\alpha, \beta, \|a\|, \|b\|)\,(\|F\|_{V'} + \|G\|_{Q'}).$$

Baby picture: Mixed well-posedness needs coercivity *only on the constraint kernel*, plus inf-sup compatibility between the two spaces.

106. Stokes Flow Overview

Strong form. For viscous incompressible flow at low Reynolds:

$$-\mu\,\Delta u + \nabla p = f \text{ in } \Omega, \qquad \nabla\cdot u = 0 \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega.$$

Weak form. Find $(u, p) \in V \times Q$ with

$$a(u, v) + b(v, p) = F(v), \qquad b(u, q) = 0,$$

where $a(u, v) = \mu \int_\Omega \nabla u : \nabla v$, $b(v, q) = -\int_\Omega q\,\nabla\cdot v$, $V = H_0^1(\Omega)^n$, $Q = L_0^2(\Omega) := \{q \in L^2 : \int q = 0\}$.
Stable mixed elements. Taylor–Hood P_2–P_1, mini element, Crouzeix–Raviart, Scott–Vogelius (on barycentric refinement).
Baby picture: Stokes is the prototype mixed problem: divergence-free velocity with pressure as Lagrange multiplier for incompressibility.

C. Final Memory Sheet

The Most Important Ideas

1. A **basis** is the minimal construction kit for a vector space.
2. A **direct sum** gives a unique decomposition of vectors.
3. A **norm** defines distance, convergence, and topology.
4. **Equivalent norms** give the same topology.
5. **Completeness** means Cauchy sequences converge inside.
6. A **Banach space** is a complete normed space.
7. A **Hilbert space** is a complete inner-product space.
8. **Projection** gives best approximation in Hilbert spaces.
9. A **linear operator** is the infinite-dimensional version of a matrix.
10. **Solving** means inverting an operator.
11. **Well-posedness** means existence, uniqueness, and stability.
12. **Functionals** are continuous sensors on vector spaces.
13. **Riesz** identifies Hilbert spaces with their duals.
14. **Weak convergence** means all sensors converge.
15. L^p spaces identify functions that agree almost everywhere.
16. **Weak derivatives** are derivatives detected by integration by parts.
17. H^1 means u and ∇u both live in L^2.
18. H_0^1 means zero boundary condition in the trace/closure sense.
19. **Poincaré inequality** lets the gradient control the function on H_0^1.
20. **Sobolev spaces** are the correct spaces for weak PDE and FEM.
21. **Riesz's theorem**: a closed bounded unit ball is compact iff the space is finite-dimensional.
22. **Hahn–Banach** gives norm-preserving functional extensions and separation of convex sets.
23. **Big Four**: Open Mapping, Bounded Inverse, Closed Graph, and Banach–Steinhaus all come from Baire category.
24. **Reflexivity and Banach–Alaoglu**: weak-* compactness of dual unit balls is the existence engine.
25. **Riesz L^p duality**: $(L^p)' \cong L^q$ for Hölder-conjugate exponents, $1 \le p < \infty$.

26. **Sobolev embedding** has three regimes governed by mp vs n: subcritical (into L^q), critical (into all finite L^q), supercritical (into Hölder continuous).
27. **Rellich–Kondrachov**: $H^1 \hookrightarrow\hookrightarrow L^2$ on bounded Lipschitz domains — compactness for existence.
28. **Lax–Milgram** (continuity M, coercivity α) gives existence with stability $\|u\| \le \alpha^{-1}\|F\|$.
29. **BNB** replaces coercivity by inf-sup compatibility for non-symmetric trial-test pairings.
30. **Céa's lemma**: Galerkin is quasi-optimal: $\|u - u_h\|_V \le (M/\alpha) \inf_{v_h \in S_h} \|u - v_h\|_V$.
31. **Aubin–Nitsche**: dual-problem trick buys one extra power of h in L^2.
32. **Ciarlet element** (K, P_K, Σ_K): geometry + local space + DoFs; unisolvence makes assembly possible.
33. **Korn's inequality** is the elasticity coercivity engine: symmetric gradient controls full gradient on H^1_0.
34. **Inf-sup (Babuška–Brezzi)** is the well-posedness criterion for mixed (u, p) problems — equal-order spaces are unstable.
35. **Green's formula in** H^1: prove by continuity of three bilinear pieces, then density from $C^1(\overline{\Omega})$.
36. **Bounded bilinear** $\Leftrightarrow$ **continuous**: $|a(u, v)| \le M\|u\|\|v\|$ is the standard continuity criterion (HW5).
37. **Compact embedding** $X \subset\subset Y$: continuous inclusion + bounded sets map to precompact sets; $H^1 \subset\subset L^2$ (HW6).
38. **Non-homogeneous Dirichlet**: lift $u = \bar{u} + w$ with $w \in H^1_0$; stability tracks both f and $\bar{u}$ (HW6).
39. **Robin coercivity**: weak limit in H^1 + Rellich strong L^2 limit kills boundary mass (HW6).
40. P_1 **FEM density**: $\bigcup_n V_n$ is dense in L^2 and H^1 on $(0, 1)$ (HW7).

Most Important Traps

1. Span and linear independence are different: span is about building; independence is about redundancy.
2. $V = X + Y$ is not enough for a direct sum; you also need $X \cap Y = \{0\}$.
3. Existence, uniqueness, and stability are different requirements.
4. Injective means uniqueness; surjective means existence.
5. A bijective operator is not enough for well-posedness unless the inverse is continuous.
6. Cauchy does not imply convergent unless the space is complete.
7. Closed means limits stay inside, not merely that the set has a boundary.
8. In infinite dimensions, norms are not automatically equivalent.
9. Closure depends on the norm.
10. Bounded does not imply compact in infinite dimensions.
11. Projection requires Hilbert structure and closed subspaces.
12. Weak convergence is not strong convergence.
13. L^p functions are equivalence classes, not pointwise objects.
14. A weak derivative is not necessarily a classical pointwise derivative.
15. $\|\nabla u\|_{L^2}$ is not a norm on all of H^1; constants break it.
16. Poincaré inequality needs boundary/mean-zero-type restrictions.
17. Boundary values of Sobolev functions require trace theory.
18. FEM is not just matrix algebra; it is projection in a function space.
19. Continuity of a and coercivity of a are independent: you need both for Lax–Milgram.

20. Sobolev embedding has three regimes; check mp vs n before claiming a continuous representative.
21. Pure Neumann problems require the compatibility condition $\int_\Omega f + \int_{\partial\Omega} h = 0$.
22. Essential BCs are imposed via the space; natural BCs come from integration by parts in the functional.
23. Equal-order (P_1, P_1) velocity–pressure pairs violate discrete inf-sup; use Taylor–Hood or a stabilized scheme.
24. Korn's inequality requires either a Dirichlet portion or a quotient by rigid-body motions.
25. L^1 and L^∞ are not reflexive; do not use Banach–Alaoglu sequential extraction in L^1.
26. Density of $\mathcal{D}(\Omega)$ in L^p fails at $p = \infty$.
27. Inverse inequalities are FEM-only: they hold inside V_h, not for general functions in V.
28. A step function can lie in L^1 without a weak derivative in L^1; $1 - |x|$ is the positive corner example.
29. Edge-midpoint DoFs on Q_1 squares fail unisolvence; Hermite triangles need not give global C^1 conformity.
30. Nodal-spike FEM sequences can converge in L^p but fail to be Cauchy in H^1.

One-Line Course Summary

$$\boxed{\text{Linear algebra} \Rightarrow \text{operator equations} \Rightarrow \text{well-posedness} \Rightarrow \text{weak PDE} \Rightarrow \text{FEM projection.}}$$

$$\boxed{\text{Choose the right space} \Rightarrow \text{prove the weak solution exists} \Rightarrow \text{approximate it by projection in FEM.}}$$

$$\boxed{\text{Norm} = \text{ruler}, \quad \text{Banach} = \text{no holes}, \quad \text{Hilbert} = \text{geometry}, \quad \text{Sobolev} = \text{PDE-ready.}}$$

$$\boxed{H_0^1 \text{ is the key FEM energy space: } u, \nabla u \in L^2, \quad u|_{\partial\Omega} = 0, \quad \|u\|_{H^1} \sim \|\nabla u\|_{L^2}.}$$

Final Additions from HW and Late-Course Material

- $C^1(\overline{\Omega})$ can be an inner-product space but still fail to be Hilbert; the arctan/log sequence is a concrete Cauchy witness.
- Green's formula and bounded-bilinear continuity are proved by density plus $|a(u,v)| \le M\|u\|\|v\|$.
- Compact embedding $H^1 \subset\subset L^2$ drives Robin coercivity and Fredholm-type arguments (HW6).
- Non-homogeneous Dirichlet: lift to H_0^1; Robin boundary terms in the bilinear form (HW6).
- Union of P_1 spaces is dense in H^1; unisolvence and global C^1 conformity are not automatic (HW7).
- S_h is finite-dimensional, hence complete; if exact u exists, FEM gives energy-orthogonal best approximation.