# Goal-Oriented Projection-Based Model Order Reduction

Zahm–Billaud-Friess–Nouy (2017) • 2D heat example

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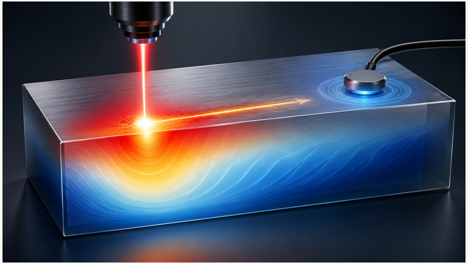
# Motivation

- Physics: a heat source moves across the part; material or load varies with $\xi$.
- What we want: one number $s(\xi)$ at the probe — not the full temperature field.
- Challenge: each new $\xi$ means a new expensive simulation $\Rightarrow$ build a fast surrogate for $s(\xi)$.



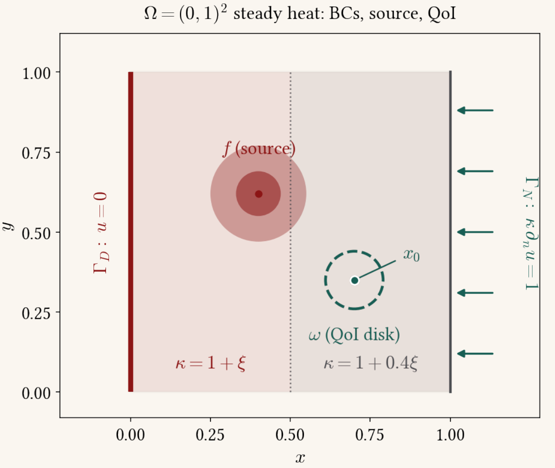
# Outline

# 2D steady heat on $\Omega = (0,1)^2$

$$-\nabla\cdot(\kappa\,\nabla u) = f, \quad u = 0 \text{ on } \Gamma_D, \quad \kappa\,\partial_n u = g \text{ on } \Gamma_N.$$

$$s = \frac{1}{|\omega|}\int_\omega u = Lu.$$

- $u$: temperature; $\kappa$: conductivity; $f$: source; $g$: boundary flux.
- $\xi$: parameter in $\kappa(x,\xi)$ (not a spatial coordinate).
- $s$: disk average at probe $\omega$ (bounded on $H^1$; point value is not).




GB Part 7, §5.1 [continuous elliptic], GB Part 4, Def. 1.68 [bounded $L$]

# Weak form

$$\int_\Omega \kappa \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x + \int_{\Gamma_N} g\, v \,\mathrm{d}s \qquad \forall v \in \mathcal{V}.$$

$$a(u, v) = \ell(v).$$

- $v$: test function; integration by parts moves derivatives onto $v$.
- $\mathcal{S} = \mathcal{V} = H^1_{0,\Gamma_D}$: $v = 0$ on $\Gamma_D$; Neumann data $g$ enters $\ell(v)$.

GB Part 7, §5.1 [IBP → bilinear form], GB Part 6, (3.1) [variational (3.1)]

# Well-posedness

$$\|u\|_{L^2} \le C_P \, \|\nabla u\|_{L^2}\,, \qquad a(u,u) \ge \alpha \, \|u\|_{H^1}^2\,, \qquad \exists!\, u \in \mathcal{S}:\; a(u,v) = \ell(v) \;\forall v \in \mathcal{V}.$$

- Continuity of $a$ and $\ell$; Poincaré $\Rightarrow$ coercivity; Lax–Milgram $\Rightarrow$ unique stable $u$.

GB Part 7, Lem. 5.4 [Poincaré], GB Part 7, §5.4 [strong coercivity], GB Part 6, Cor. 3.10 [Lax–Milgram]

# Operator form

$$\langle Au, v\rangle = a(u,v), \quad \langle b, v\rangle = \ell(v), \qquad Au = b.$$

- $\langle\cdot,\cdot\rangle$: duality pairing; $A : \mathcal{S} \to \mathcal{V}'$, $b \in \mathcal{V}'$.
- $u \in \mathcal{S}$ trial; $v \in \mathcal{V}$ test; later $\mathcal{S}_h \subset \mathcal{S}$, $\mathcal{V}_h \subset \mathcal{V}$ with $u_h \in \mathcal{S}_h$, $v_h \in \mathcal{V}_h$.
- Output $s = Lu$ (demo scalar; Zahm et al. vector $Z = \mathbb{R}^l$ in supplementary slides).

GB Part 6, (3.1) [$Au = b$], GB Part 4, Def. 1.68 [bounded $A$], GB Part 4, Def. 1.73 [dual $\mathcal{V}'$], GB Part 3, Def. 1.48 [Hilbert]

# Parameter $\xi$

$$A(\xi)\,u(\xi) = b(\xi), \qquad s(\xi) = L(\xi)\,u(\xi), \qquad \xi \in \Xi.$$

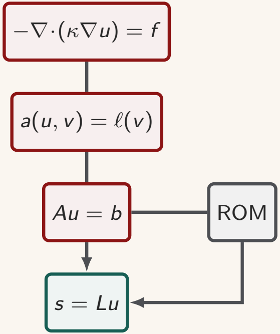
- $\xi$: scenario knob (conductivity in demo; BC/load in general) — not a spatial coordinate.
- Many queries: design / UQ need thousands of $\xi$; full FEM each time is too slow.
- ROM: **offline** snapshots at $\xi_i$; **online** cheap $s(\xi)$ per query.

GB Part 6, (3.1) [variational form], GB Part 4, Def. 1.68 [bounded parametric maps]

# Parametric output problem

$$A(\xi)\,u(\xi) = b(\xi), \qquad s(\xi) = L(\xi)\,u(\xi), \qquad \xi \in \Xi.$$

- $s(\xi)$: output of interest (sensor), not the full field $u$.
- **Offline:** build $\mathcal{S}_h, \mathcal{V}_h, W_k^Q$; **online:** $s(\xi)$ per query.




GB Part 6, (3.1) [variational form], GB Part 4, Def. 1.68 [bounded $L$]

# Adjoints $A^*$, $L^*$ and dual operator $Q$

$$\langle Au, v\rangle = \langle u, A^* v\rangle \; \forall u \in \mathcal{S},\; v \in \mathcal{V}.$$

$$A^* Q = L^*, \qquad Q \in \mathcal{L}(Z', \mathcal{V}), \qquad s = Lu = Q^* b.$$

- $A^*$: test loads $\to$ trial space; $L^*$: output component / sensor weight in $Z'$.
- $Q$: influence function in $\mathcal{V}$ — response to unit output loading (probe $\Rightarrow$ heat on $\omega$).
- Exact $u$ or exact $Q$ recovers $s$; ROM approximates both.

GB Part 4, Thm. 1.82 [Riesz], GB Part 7, Def. 5.7 [adjoint $A^*$], GB Part 6, Thm. 3.9 [well-posed dual]

# What Zahm et al. add

- **Goal-oriented:** reduced spaces target $s = Lu$, not only the field $u$.
- **Petrov–Galerkin:** $\mathcal{S}_h \neq \mathcal{V}_h$ plus dual space $W_k^Q$ (three reduced spaces).
- **Vector output:** $s \in Z$ with operator dual $Q \in \mathcal{L}(Z', \mathcal{V})$ (not only scalar QoI).
- **Certified bounds:** computable $\Delta(\xi)$ from primal and dual residuals.
- **Greedy enrichment:** sample at $\arg\max_\xi \Delta(\xi)$.
- Three estimators (pg $\to$ pd $\to$ sp): pg $= Lu_h$; pd adds dual correction; sp uses saddle point — definitions and ladder in supplementary slides.

GB Part 6, §3.3 [Ritz–Galerkin], GB Part 6, §3.4 [discrete PG], GB Part 6, Thm. 3.9 [BNB], GB Part 3, Exo. 1.42 [Cauchy–Schwarz]

# inf–sup stability

$$\alpha \|u\|_{\mathcal{S}} \le \|Au\|_{\mathcal{V}'} \le \beta \|u\|_{\mathcal{S}} ,$$

$$\alpha = \inf_{u \neq 0} \sup_{v \neq 0} \frac{\langle Au, v \rangle}{\|u\|_{\mathcal{S}} \|v\|_{\mathcal{V}}}, \quad \beta = \sup_{u} \sup_{v} \frac{\langle Au, v \rangle}{\|u\|_{\mathcal{S}} \|v\|_{\mathcal{V}}}.$$

- $\alpha > 0$: stability (BNB / inf–sup); small residual $\Rightarrow$ small error.
- Allows $\mathcal{S} \neq \mathcal{V}$ (Petrov–Galerkin); heat with $\mathcal{S} = \mathcal{V}$ is the Lax–Milgram case.

GB Part 6, Thm. 3.9 [BNB], GB Part 6, Cor. 3.10 [Lax–Milgram]

# Petrov–Galerkin projection

$$\langle Au_h - b,\ v_h\rangle = 0 \qquad \forall v_h \in \mathcal{V}_h, \qquad u_h \in \mathcal{S}_h \subset \mathcal{S}.$$

$$\|u - u_h\|_{\mathcal{S}} \le \frac{1}{\sqrt{1 - \delta^2_{\mathcal{S}_h,\mathcal{V}_h}}} \min_{u_h \in \mathcal{S}_h} \|u - u_h\|_{\mathcal{S}}\,.$$

- Residual $Au_h - b$ orthogonal to $\mathcal{V}_h$; $u_h \in \mathcal{S}_h$, $v_h \in \mathcal{V}_h$ ($\mathcal{S}_h \neq \mathcal{V}_h$ in general).
- $\delta_{\mathcal{S}_h,\mathcal{V}_h} < 1$ measures test-space quality (Prop. 2.1).

GB Part 6, §3.4 [discrete PG, eq. (3.9)], GB Part 6, (3.10) [discrete inf–sup $m_h$], GB Part 6, Lem. 3.11 [Céa], GB Part 6, §3.3 [Ritz–Galerkin]

# Output error bound (Prop. 2.1, Eq. 9)

$$\|s - Lu_h\|_Z \leq \delta^L_{\mathcal{V}_h} \cdot \frac{1}{\sqrt{1-\delta^2_{\mathcal{S}_h,\mathcal{V}_h}}} \cdot \min_{u_h \in \mathcal{S}_h} \|u - u_h\|_{\mathcal{S}} \, .$$

- Compliant case ($\mathcal{S} = \mathcal{V}$, $\mathcal{V}_h = \mathcal{S}_h$, $Lu = \langle b, u \rangle$): $|s - Lu_h| \leq \|u - u_h\|^2_{\mathcal{S}}$.

GB Part 6, Lem. 3.11 [Céa quasi-opt.], GB Part 3, Exo. 1.42 [Cauchy–Schwarz], GB Part 7, §5.2 [Ritz–Galerkin]

# Two jobs of $\mathcal{V}_h$

$$\|s - Lu_h\|_Z \le \underbrace{\delta^L_{\mathcal{V}_h}}_{\text{(c) dual range}} \underbrace{\frac{1}{\sqrt{1-\delta^2_{\mathcal{S}_h,\mathcal{V}_h}}}}_{\text{(b) PG quality}} \underbrace{\min_{u_h \in \mathcal{S}_h} \|u - u_h\|_{\mathcal{S}}}_{\text{(a) trial}}.$$

- $\mathcal{V}_h$ tests the projection **and** approximates $\mathrm{range}(A^* L^*)$.
- Demo uses shared snapshot space; Zahm et al. allow $\mathcal{V}_h \neq \mathcal{S}_h$.

GB Part 6, §3.4 [discrete test spaces], GB Part 6, Thm. 3.9 [inf-sup stability]

# Dual problem

$$A^*Q = L^*, \qquad s = Lu = Q^*Au = Q^*b.$$

- $Q$: dual / influence function — which test directions matter for $s$?
- For probe average: $Q$ is the response to a unit load on $\omega$.
- Either exact $u$ or exact $Q$ recovers $s$; ROM approximates both.

GB Part 4, Thm. 1.82 [Riesz], GB Part 7, Def. 5.7 [adjoint $A^*$], GB Part 6, Thm. 3.9 [well-posedness]

# Primal–dual estimate

$$\tilde{s} = Lu_h + Q_h^* (b - Au_h),$$
$$\|s - \tilde{s}\|_Z \leq \|u - u_h\|_{\mathcal{S}} \; \|L^* - A^* Q_h\|_{Z' \to \mathcal{S}'} .$$

- $\tilde{s}$: field estimate + correction from primal residual $b - Au_h$.
- Three reduced spaces: trial $\mathcal{S}_h$, test $\mathcal{V}_h$, dual $W_k^Q$.
- Zahm et al. also give a saddle-point estimator (Prop. 2.10) for a tighter bound ladder — supplementary slides.

GB Part 3, Exo. 1.42 [C–S on pairing], GB Part 4, Thm. 1.82 [Riesz], GB Part 7, Def. 5.7 [adjoint], GB Part 3, Def. 1.51 [orthogonal projection]

# Certified bound

$$\|s(\xi) - \tilde{s}(\xi)\|_Z \leq \frac{\|A(\xi)u_h - b(\xi)\|_{\mathcal{S}_0'} \ \|L^*(\xi) - A^*(\xi)Q_h(\xi)\|_{Z' \to \mathcal{S}_0'}}{\alpha(\xi)} =: \Delta(\xi).$$

- $\Delta(\xi)$: computable upper bound (primal residual $\times$ dual residual / $\alpha$).
- Certified: true error $\leq \Delta$; effectivity $\Delta / \|s - \tilde{s}\| \geq 1$.

GB Part 6, (3.5) [inf-sup const. $m$], GB Part 6, Thm. 3.9 [BNB stability], GB Part 3, Exo. 1.42 [residual product]

# Greedy enrichment

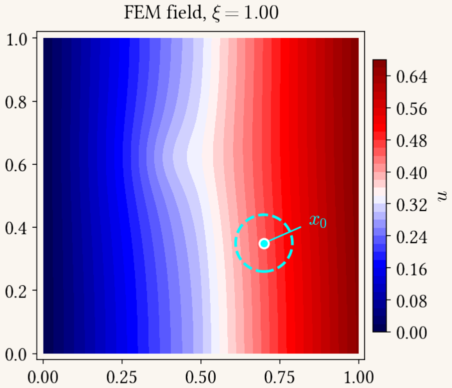
$$\xi^\star \in \arg\max_{\xi\in\Xi} \Delta(\xi), \qquad \mathcal{S}_h \leftarrow \mathcal{S}_h + \operatorname{span} u(\xi^\star), \quad W_k^Q \leftarrow W_k^Q + \operatorname{range} Q(\xi^\star).$$

- Sample where the bound is largest; solve full $u, Q$ at $\xi^\star$; enrich bases.
- Repeat until $\max_\xi \Delta(\xi)$ is below tolerance.

Alternate / simultaneous enrichment (Alg. 1–2); $\mathcal{V}_h$ from preconditioner $P_m(\xi) \approx A(\xi)^{-1}$ (Eq. 44). GB Part 4, Thm. 1.83 [weak compactness], GB Part 6, Thm. 3.9 [certified $\alpha$]

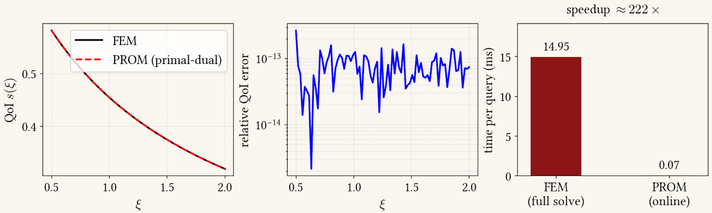
# FEM field at $\xi = 1$

- Full FEM; QoI $s = \bar{u}_\omega$ read from the field.
- PROM predicts $s(\xi)$ for many $\xi$ at reduced cost.

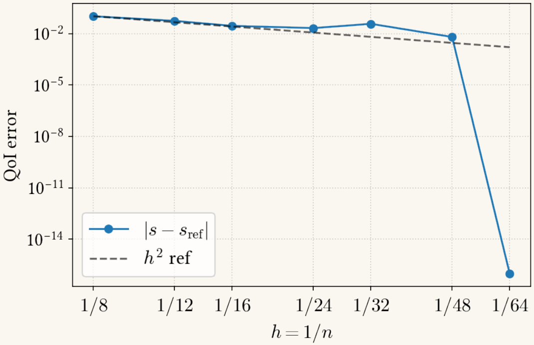


# PROM vs FEM: online speedup

- 80 parametric queries, $\xi \in [0.5, 2]$.
- PROM: primal–dual, $R = 10$ snapshots.
- $\sim$250–370$\times$ faster per online query; QoI error $\sim 10^{-13}$.



# Demo scope




- **Implemented:** parametric $A(\xi)$ (affine in demo), primal–dual online $s_{\mathrm{pd}}$.
- **Zahm et al. add:** separate $\mathcal{V}_h$, saddle point, greedy $\Delta(\xi)$ — not in demo.
- Left: FEM mesh convergence ($h^2$ QoI rate).

# Takeaways

1. Field $u$ is a means; vector output $s = Lu$ + certified $\Delta(\xi)$ is the goal.
2. Weak form $\to Au = b$; inf–sup gives stability and certified bounds.
3. Primal–dual estimate + certified $\Delta(\xi)$; demo uses pd only (saddle point: supplementary).
4. Demo: primal–dual step only; full FEM vs PROM — large online speedup.

GB Part 6, §3.3 [Galerkin], GB Part 6, §3.4 [PG], GB Part 6, Thm. 3.9 [BNB], GB Part 3, Exo. 1.42 [C–S], GB Part 4, Thm. 1.82 [Riesz]

# Reference

O. Zahm, M. Billaud-Friess, A. Nouy, *SIAM J. Sci. Comput.* **39**(4), A1647–A1674 (2017).

**Thank you.**

Special thanks to José Hasbani & Obed Camacho for discussions.

# Supplementary slides

- Standard Galerkin ROM baseline and comparison table.
- Affine parametrization (demo detail).
- Full board proof sketches for Props. 2.4 and 2.5.
- Three estimators (pg, pd, sp), saddle-point system, and error-bound ladder (Prop. 2.10).
- Output space $Z$, dual $Z'$, and vector-valued outputs.

# Output space $Z$, dual $Z'$, and adjoint $L^*$

$$L : \mathcal{S} \to Z, \qquad s = Lu \in Z.$$

$$\langle L^* z', u\rangle = \langle z', Lu\rangle \ \forall z' \in Z',\ u \in \mathcal{S}.$$

- $Z$: output values; demo $Z = \mathbb{R}$, Zahm et al. $Z = \mathbb{R}^l$ or trace space.
- $\mathcal{S}', \mathcal{V}', Z'$: duals via $\langle \cdot, \cdot \rangle$ (Riesz on Hilbert spaces).
- $L^*$: sensor weight in $Z'$; pairs with $Q \in \mathcal{L}(Z', \mathcal{V})$ in $A^* Q = L^*$.

GB Part 4, Def. 1.73 [dual space], GB Part 4, Def. 1.68 [bounded $L$], GB Part 7, Def. 5.7 [adjoint $L^*$]

# Vector output and operator dual

$$s = Lu \in Z, \qquad L \in \mathcal{L}(\mathcal{S}, Z),$$
$$A^* Q = L^*, \qquad Q \in \mathcal{L}(Z', \mathcal{V}).$$

- Demo: $Z = \mathbb{R}$, $s = \bar{u}_\omega$; Zahm et al.: $Z = \mathbb{R}^l$ or trace space.
- **Novelty:** one reduced $W_k^Q$ for all components of $s$.

GB Part 4, Def. 1.68 [bounded $L$], GB Part 4, Thm. 1.82 [Riesz], GB Part 7, Def. 5.7 [adjoint]

# Standard Galerkin ROM (baseline)

- **Offline:** full solves at sample parameters $\xi_i$; build $\mathcal{S}_h = \mathrm{span}\{u(\xi_i)\}$ (POD/RB).
- **Online:** find $u_h \in \mathcal{S}_h$ by Galerkin projection, $\langle Au_h - b, v_h \rangle = 0 \ \forall v_h \in \mathcal{V}_h = \mathcal{S}_h$.
- Focus: approximate $u$ in an energy norm; output $s = Lu$ is evaluated post hoc.
- Not the same as full-order discrete Galerkin FEM — this is a second reduction on top of FEM.

# Paper vs standard Galerkin ROM

| Standard Galerkin ROM | This paper |
|---|---|
| reduce $u$ | **goal-oriented** $s = Lu$ |
| $\mathcal{S}_h = \mathcal{V}_h$ | **Petrov–Galerkin** $\mathcal{S}_h \neq \mathcal{V}_h$ |
| scalar / compliant QoI | **vector** $s \in Z$, operator $Q$ |
| heuristic output error | **certified** $\Delta(\xi)$ |
| fixed basis | **greedy** at $\arg\max \Delta$ |

- Three estimators: Petrov–Galerkin $\to$ primal–dual $\to$ saddle point (provable error ladder).

# Affine parametrization

$$\kappa(x,\xi) = 1 + \xi\beta(x), \qquad A(\xi) = A_0 + \xi A_1, \qquad A(\xi)u(\xi) = b(\xi), \quad s(\xi) = Lu.$$

- Example (demo): $\beta = 1$ for $x < 0.5$, $\beta = 0.4$ for $x \geq 0.5$.
- Affine $A(\xi) = A_0 + \sum_k \theta_k(\xi) A_k$ matches FEM assembly: online cost is reassembly, not full mesh build.
- This is a **demo / implementation** choice (heat conductivity), not a hypothesis in Zahm et al. (2017).

GB Part 6, (3.1) [variational form], GB Part 4, Def. 1.68 [affine operators]

# Parameter $\xi$ (with affine example)

$$\kappa(x,\xi) = 1 + \xi\beta(x), \qquad A(\xi) = A_0 + \xi A_1, \qquad \xi \in \Xi.$$

- $\xi$: scenario knob (conductivity in demo; BC/load in general) — not a spatial coordinate.
- Many queries: design / UQ need thousands of $\xi$; full FEM each time is too slow.
- ROM: **offline** snapshots at $\xi_i$; **online** cheap $s(\xi)$ per query.

GB Part 6, (3.1) [variational form], GB Part 4, Def. 1.68 [affine operators]

# Board: squared effect (Prop. 2.4) — full sketch

**Compliant case:** $\mathcal{S} = \mathcal{V}$, $R_{\mathcal{S}} = A$, $\mathcal{V}_h = \mathcal{S}_h$, $Lu = \langle b, u \rangle$.

$$|s - Lu_h| \leq \|u - u_h\|_{\mathcal{S}}^2 .$$

- $s - Lu_h = \langle b, u - u_h \rangle$; Galerkin: $\langle b, u_h - u' \rangle = 0$ for $u' \in \mathcal{S}_h$.
- Cauchy–Schwarz: $|\langle b, u - u_h \rangle| \leq \|u - u_h\|_{\mathcal{S}} \|b - Au_h\|_{\mathcal{S}'} = 0$ first order; sharper bound uses energy orthogonality $\Rightarrow$ square.

# Board: primal–dual error (Prop. 2.5) — full sketch

$$s - \tilde{s} = L(u - u_h) - Q_h^* A(u - u_h) = (L - A^* Q_h)(u - u_h) \text{ in pairing sense.}$$

- Apply Cauchy–Schwarz on dual pairing $\Rightarrow$ product bound.
- Prop. 2.8: build $Q_h$ by projection in $W_k^Q$ for sharper dual factor.

# Three estimators: pg, pd, sp

- **pg (Petrov–Galerkin):** find $u_h \in \mathcal{S}_h$ with $\langle Au_h - b, v_h \rangle = 0 \ \forall v_h \in \mathcal{V}_h$; output $\tilde{s}_{\mathrm{pg}} = Lu_h$ (field estimate only).
- **pd (primal–dual):** same $u_h$ plus $Q_h \in W_k^Q$; $\tilde{s}_{\mathrm{pd}} = Lu_h + Q_h^*(b - Au_h)$ (Prop. 2.5; demo uses this online).
- **sp (saddle point):** mixed solve on $T_p = \mathcal{V}_h + W_k^Q$ (Prop. 2.10); $\tilde{s}_{\mathrm{sp}}$ from one more Galerkin orthogonality — tightest bound, not in demo.

GB Part 6, §3.4 [discrete PG], GB Part 6, §3.3 [Galerkin orthog.], GB Part 3, Exo. 1.42 [Cauchy–Schwarz]

# Saddle-point system (Prop. 2.10)

$$R_{\mathcal{V}} = AR_{\mathcal{S}}^{-1}A^*, \qquad T_p = \mathcal{V}_h + W_k^Q.$$

$$\langle R_{\mathcal{V}} v_{h,p}, v_h\rangle + \langle Au_{h,p}, v_h\rangle = \langle b, v_h\rangle \ \forall v_h \in T_p, \qquad \langle A^* v_{h,p}, u\rangle = 0 \ \forall u \in \mathcal{S}_h.$$

- $R_{\mathcal{V}}$: energy-weighted test norm; mixed multiplier $v_{h,p}$.
- One more orthogonality on $T_p \Rightarrow$ best bound in the pg/pd/sp ladder.
- Not used in the heat demo; included for certified bound bookkeeping in Zahm et al.

GB Part 6, §3.4 [discrete PG, eq. (3.9)], GB Part 6, Thm. 3.9 [BNB], GB Part 4, Thm. 1.82 [Riesz]

# Error ladder (pg $\to$ pd $\to$ sp)

$$\|s - \tilde{s}_{\mathrm{sp}}\|_Z \le \|s - \tilde{s}_{\mathrm{pd}}\|_Z \le \|s - \tilde{s}_{\mathrm{pg}}\|_Z \,.$$

- Petrov–Galerkin, then primal–dual, then saddle point: each step adds Galerkin orthogonality.
- Bounds tighten; online reduced-system size and cost grow.

GB Part 6, §3.3 [Galerkin orthog.], GB Part 3, Exo. 1.42 [C–S], GB Part 3, Def. 1.51 [orthogonal projection]