

Mechanics: Elasticity and Inelasticity

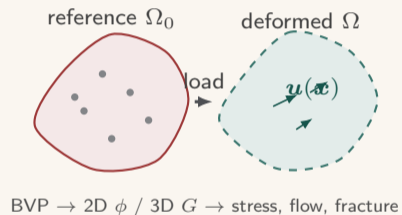
A brief intro • ME 340

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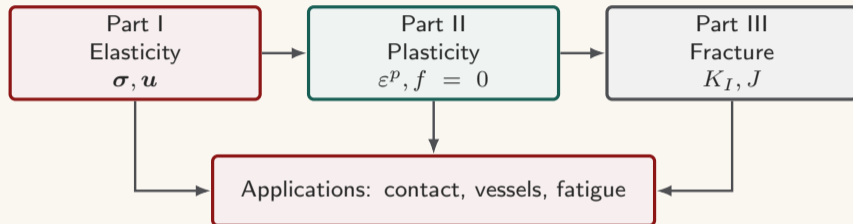
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Spring 2026

- ME 340 (Wei Cai): elasticity, plasticity, fracture.
- I—2D ϕ , 3D G , contact; II—yield, flow, hardening; III—LEFM, J , fatigue.
- Analytic + Matlab.
- Barber (2010); Anderson (2005).
- Notes ● PDF.



Three-part course structure



I. stress, strain, equilibrium

II. yield, flow, hardening

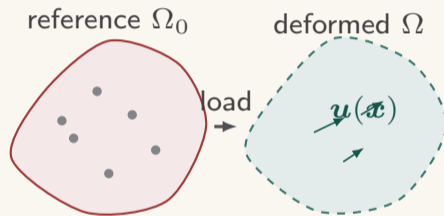
III. crack-tip fields, energy release

Core idea: well-posed BVP \rightarrow elastic solution \rightarrow plasticity if $f > 0 \rightarrow$ fracture if cracks grow.

Outline

- 1 Foundations
- 2 One-dimensional and rod problems
- 3 Two-dimensional elasticity
- 4 Three-dimensional elasticity
- 5 Plasticity
- 6 Fracture mechanics
- 7 Problem-solving workflow

Continuum body: reference and deformed configurations



- Ω_0 : reference configuration; Ω : deformed configuration.
- Material points
 $\mathbf{X} \mapsto \mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X})$.
- Small strain: $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.
- The “potato” is any bounded body V — geometry is arbitrary.

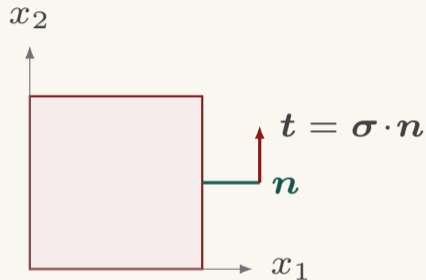
Lec. 1 [Introduction]

Stress and strain

$$\sigma_{ij} = \sigma_{ji}, \quad \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

- σ : Cauchy stress; ε : small strain; u : displacement.
- Traction on plane normal \mathbf{n} : $t_i = \sigma_{ij}n_j$.
- Static equilibrium: $\partial\sigma_{ij}/\partial x_j = 0$ (no body force).

Lec. 1–2 [Introduction, Tensors]



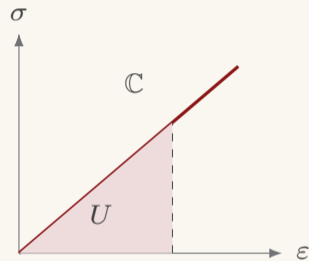
Cauchy cube; traction $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ on a cut plane

Hooke's law and elastic energy

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad U = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} dV.$$

- Isotropic: $\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$.
- E , ν , Lamé λ , μ ; positive-definite \mathbb{C} .
- Stored energy = area under linear σ - ε law.

Lec. 3 [Hooke's Law], Lec. 4 [Fundamental Equations]



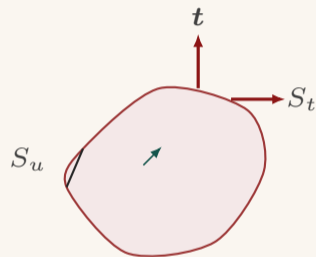
shaded area = strain-energy density

Fundamental boundary-value problem

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top), \quad \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}, \quad \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \text{ in } V.$$

- $\mathbf{u} = \mathbf{u}_0$ on S_u ; $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}$ on S_t .
- Compatibility if $\boldsymbol{\varepsilon}$ is not from a single \mathbf{u} .

Lec. 4 [Fundamental Equations]



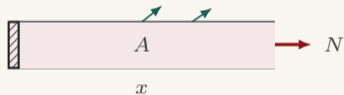
body V , boundary $\partial V = S_u \cup S_t$

Elastic rod (1D building block)

$$\varepsilon = \frac{du}{dx}, \quad N = EA\varepsilon, \quad \frac{dN}{dx} + f = 0.$$

- Axial bar: $u(x)$, stress $\sigma = N/A$, rigidity EA .
- Same pattern as 3D theory, one displacement component.
- Warm-up before 2D Airy and 3D Green's functions.

Lec. 4–5 [Fund. Eqs., 2D Elasticity]



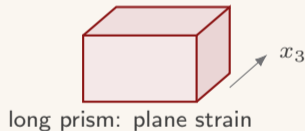
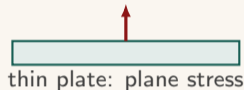
fixed end + axial load \Rightarrow extension $u(x)$

2D formulations

- **Plane strain:** $\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$ (thick body).
- **Plane stress:** $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ (thin plate).
- Airy $\phi(x, y)$ with $\nabla^4 \phi = 0$:

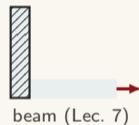
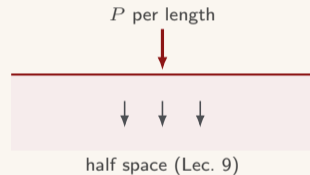
$$\sigma_{xx} = \phi_{,yy}, \quad \sigma_{yy} = \phi_{,xx}, \quad \sigma_{xy} = -\phi_{,xy}.$$

Lec. 5 [2D Elasticity]



Classic 2D solution routes

- **Rectangular beam** (Lec. 7): polynomial Airy + BC matching.
- **Fourier series** (Lec. 8): periodic / strip loads.
- **Half space** (Lec. 9): Flamant line load; $\sigma \sim 1/r$.
- **Contact** (Lec. 10): unknown contact patch and pressure.

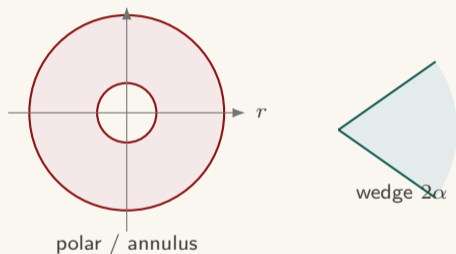


Polar coordinates and wedge problems

$\nabla^4 \phi = 0 \Rightarrow \phi(r, \theta)$ with $r, \theta, \ln r, \theta \ln r$ modes.

- Lec. 11–12: disks, rings, holes in (r, θ) .
- Lec. 13 wedge: corner singularities; eigenfunction exponents.
- Pick modes with symmetry and finite energy.

Lec. 11–12 [Polar Coordinates, Wedge and Notch]

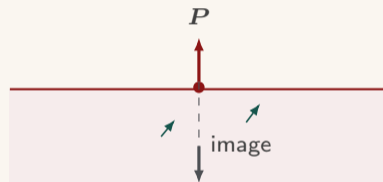


Green's function approach (3D)

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}, \quad \mathbf{u}(\mathbf{x}) = \int G(\mathbf{x}, \boldsymbol{\xi}) \mathbf{f}(\boldsymbol{\xi}) dV_{\boldsymbol{\xi}}.$$

- Kelvin solution (Lec. 17): point force in infinite space.
- Half-space images (Lec. 16): traction-free surface.
- Superpose for defects and boundaries.

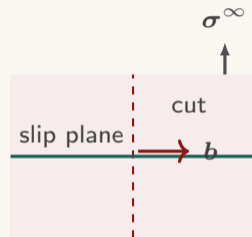
Lec. 9–10 [Half Space, Contact]



Kelvin field + surface image

Dislocations and defect fields

- Cut the lattice; insert \mathbf{b} (Burgers vector) across the slip plane.
- Stress singular at the core; far field set by \mathbf{b} and geometry.
- Peach–Köhler: defect force from external σ on the dislocation.
- Superpose: singular field + image / boundary correction.
- Macroscopic plastic strain accumulates from many dislocation motions.



Volterra cut + Burgers \mathbf{b} + far-field σ^∞

Lec. notes [Dislocations (extended notes)]

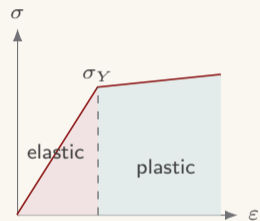
From elasticity to plasticity

- **Elastic:** load removal \Rightarrow strain returns to zero.
- **Plastic:** permanent strain remains after unloading.
- Additive split (small strain):

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p.$$

- Elastic part: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^e$.
- Plastic part: governed by a yield condition + flow rule.

Lec. 3 [Hooke's Law], Lec. 13 [Fund. Eqs. of Plasticity]



loading past σ_Y leaves permanent ε^p on unload

Yield criteria: von Mises and Tresca

Deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$.

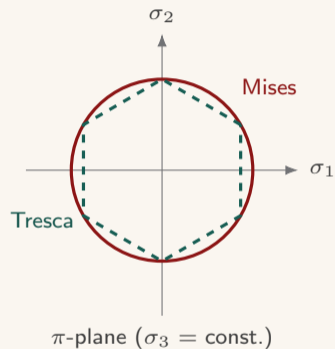
$$J_2 = \frac{1}{2}s_{ij}s_{ij}, \quad \sigma_{\text{eq}} = \sqrt{3J_2}.$$

von Mises yield:

$$f = J_2 - k^2 \leq 0, \quad k = \frac{\sigma_Y}{\sqrt{3}}.$$

Tresca yield: $\max |\sigma_i - \sigma_j| = 2k$ (alternative for metals). Plastic flow is insensitive to hydrostatic stress; yielding depends on s_{ij} .

Lec. 13–14 [Yield surface / graphical]



von Mises fits polycrystals (Taylor–Quinney)

J_2 associated flow rule

Associated (normality) flow with $f = J_2 - k^2$:

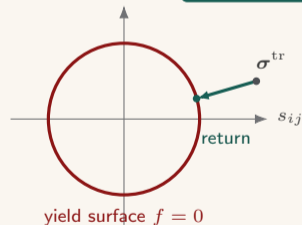
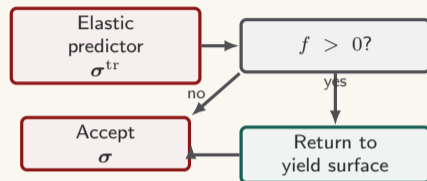
$$\dot{\epsilon}_{ij}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda} s_{ij}, \quad \dot{\epsilon}_{kk}^p = 0.$$

Consistency during plastic loading ($\dot{f} = 0$):

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p, \quad f = J_2 - k^2 = 0.$$

Incremental update: elastic predictor, plastic corrector if $f > 0$. Dislocations are the microscopic carriers; J_2 plasticity is the continuum limit.

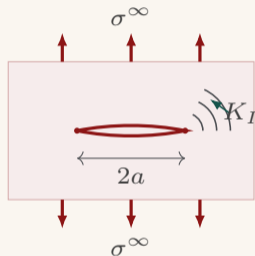
Lec. 13–15 [Flow rule, tension & shear]



Fracture: LEFM and energy release

- Slit-like crack: elastic field with $r^{-1/2}$ stress singularity at tip.
- Stress intensity K_I characterizes tip loading; fracture when $K_I = K_{Ic}$.
- Energy release rate $\mathcal{G} = \partial U / \partial a$; J -integral for nonlinear paths.
- Elastic–plastic zone and Dugdale–Barenblatt model; fatigue crack growth (Part III).

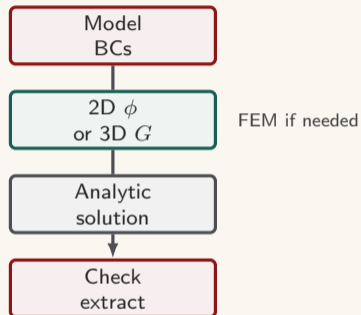
Lec. 22–26 [Slit crack, LEFM, fatigue]



center crack $2a$, remote tension σ^∞ (Mode I)

Standard workflow

- 1 **Model:** geometry, BCs, 2D vs. 3D.
- 2 **Method:** Airy ϕ or Green / images.
- 3 **Solve:** separated variables, Fourier, multipoles.
- 4 **Check:** equilibrium, BCs, finite energy.
- 5 **Extract:** σ , contact pressure, U , forces.



Benchmark mindset

Keep an analytic case as a reference when switching to numerics.

Lecture map (handouts)

Part I. Elasticity

1. Introduction
2. Tensors
3. Hooke's Law
4. Fundamental Equations
5. 2D Elasticity
6. Rectangular Beam
7. Fourier Series and Transform
8. Fourier Solution
9. Half Space
10. Contact
11. Polar Coordinates
12. Wedge and Notch

Part II. Plasticity

13. Fundamental Equations of Plasticity
14. Graphical Representations
15. Tension and Shear
16. Plastic Bending
18. Hardening Law
20. Crystal Plasticity

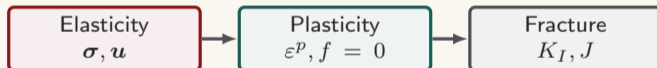
Part III. Fracture

22. Slit-like Crack
23. Energy Release Rate
24. Linear Elastic Fracture Mechanics
25. Elastic Plastic Fracture Mechanics
26. Fatigue

Cai ME 340 notes; consolidated PDF

Takeaways

- 1 **Elasticity:** kinematics + Hooke + equilibrium + BCs; 2D Airy ϕ , 3D Green's functions.
- 2 **Plasticity:** $\varepsilon = \varepsilon^e + \varepsilon^p$; von Mises yield + associated J_2 flow.
- 3 **Fracture:** K_I , \mathcal{G} , and J link elastic fields to crack growth and fatigue.
- 4 Matlab and analytic benchmarks support each part; see **ME 340 handouts**.



- W. Cai, *ME 340 Elasticity and Inelasticity*: [course page](#); [consolidated PDF](#).
- J. R. Barber, *Elasticity*, 3rd ed., Springer (2010).
- T. L. Anderson, *Fracture Mechanics*, 3rd ed., Taylor & Francis (2005).

Thank you.